algebra: $\mathcal{F}\subseteq\mathcal{P}(\Omega)$, $\emptyset\in\mathcal{F}$, $E\in\mathcal{F}\Rightarrow E^C=\Omega\backslash E\in\mathcal{F}$, $A,B\in\mathcal{F}\Rightarrow A\cup B\in\mathcal{F}$ **σ-alg**: above + $E_n\in\mathcal{F}$ for $n=1,..\Rightarrow\bigcup_1^\infty E_n\in\mathcal{F}\Rightarrow\bigcap_1^\infty E_n\in\mathcal{F}$ (countable) $\bigcap$ of σ-algs is σ-alg **Borel σ-alg**: gen. by *open sets*
Product space of $(\Omega_i,\mathcal{F}_i)_{i\in I}$: $\prod_{i\in I}\Omega_i$ (cartesian) & $\times_{i\in I}\mathcal{F}_i:=\sigma(\{A=\prod_{i\in I}A_i:A_i\in\mathcal{F}_i,\forall\text{but inf } i\, A_i=\Omega_i\})$ (A's = cylinder sets) **π-sys**: $A,B\in\mathcal{A}\Rightarrow A\cap B\in\mathcal{A}$. **$\lambda$-sys**: $\Omega\in\mathcal{A}$, $A,B\in\mathcal{A}, A\subseteq B\Rightarrow B\backslash A\in\mathcal{A}$, $A_n\subseteq\mathcal{A}$, $A_n\subseteq A_{n+1}\Rightarrow\bigcup_{n\geq1}A_n\in\mathcal{A}$ — $\mathcal{A}$ a σ-alg $\iff$ a π-sys & a λ-sys. [$B_n=\bigcup_{k=1}^n A_k$ using π-sys, as finite]
$\pi-\lambda$: $\mathcal{M}=\lambda$-s, $\mathcal{A}=\pi$-s, $\mathcal{A}\subseteq\mathcal{M}\Rightarrow\sigma(\mathcal{A})\subseteq\mathcal{M}$ [$\lambda(\mathcal{A}):=\bigcap\lambda$-s cont. $\mathcal{A}$, $\lambda(\mathcal{A})\subseteq\mathcal{M}$, $\lambda(\mathcal{A})\,\pi$-s: 2x$E=\{A\cap C\}$]
$f:\Omega\to E$ is a **ms func/RV** if $\forall A\in\mathcal{E}\, f^{-1}(A)=\{\omega\in\Omega:f(\omega)\in A\}\in\mathcal{F}$,
$\sigma(f_i:i\in I):=$ the smallest σ-alg on Ω st all $f_i:\Omega\to E_i$ are ms wrt to it.
$\sigma(X)=\{X^{-1}(A):A\in\mathcal{E}\}=\sigma(X^{-1}(A):A\in\mathcal{A})$ where $X:(\Omega,\mathcal{F})\to(E,\mathcal{E})$, $\mathcal{E}=\sigma(\mathcal{A})$
$X:\Omega\to E$ an RV: rv g on $(\Omega,\mathcal{F})$ $\sigma(X)$-ms $\iff g=h\circ X$ for an RV h on $(E,\mathcal{E})$. [none, $\downarrow$ none]
Monotone C: $\mathcal{H}$: class of bdd funcs $\Omega\to\mathbb{R}$, a vect spc over $\mathbb{R}$, $1\in\mathcal{H}$, $\mathcal{H}$ clsd under $\uparrow$ lims to bdd funcs, $\mathcal{C}\subseteq\mathcal{H}$ clsd under pointwise $\times\Rightarrow$all bdd $\sigma(\mathcal{C})$-ms funcs $\in\mathcal{H}$.

Measure

$\mu:\mathcal{F}\to[0,\infty]$ a **measure** on $(\Omega,\mathcal{F})$ if: **1.** $\mu(\emptyset)=0$ **2.** $\mu(\bigcup_{n=1}^\infty E_n)=\sum_{n=1}^\infty\mu(E_n)$ if E_n disjoint
$A\cap B=\emptyset\Rightarrow\mu(A\cup B)=\mu(A)+\mu(B)$ | $A\subseteq B\Rightarrow\mu(A)\leq\mu(B)$ | $\mu(A\cup B)+\mu(A\cap B)=\mu(A)+\mu(B)$
$A_n\uparrow A\Rightarrow\mu(A_n)\uparrow\mu(A)$[$D_n=A_n\backslash A_{n-1}$] | $B_n\downarrow B,\exists k:\mu(B_k)<\infty\Rightarrow\mu(B_n)\downarrow\mu(B)$ | $\mu(\bigcup_{n\geq1}A_n)\leq\sum_{n\geq1}\mu(A_n)$
σ-finite: $\exists(K_n)\in\mathcal{F}$ st $\forall n\,\mu(K_n)<\infty$ & $\bigcup_{n\geq1}K_n=\Omega$ | **abs.cont/equiv**: $\forall A:\nu(A)=0\Rightarrow/\iff\mu(A)=0$.
Uniq. ext $\mathcal{F}=\sigma(\mathcal{A})$ π-s, $\mu_1(\Omega)=\mu_2(\Omega)<\infty$, $\mu_1=\mu_2$ on $\mathcal{A}\Rightarrow\mu_1=\mu_2$ on $\mathcal{F}$. [$\{A:\mu_1=\mu_2\}$ a λ-s]
Cara: $(\Omega,\mathcal{F}=\sigma(\mathcal{A}=alg))$, $\mu_0:\mathcal{A}\to[0,\infty]$ count-add. set-f: outer ms ext. B meas: $\forall E\ \mu^*(E)=\mu^*(E\cap B)+\mu^*(E\cap B^C)$ | **dist func** $F:\mathbb{R}\to[0,1]$ **a)** F incr **b)** $\lim_{x\to\infty}F(x)=1$, $\lim_{x\to-\infty}F(x)=0$ **c)** F right-cont
$F_\mu(x)=\mu((-\infty,x])$ a dist func $\Leftarrow\mu$ a ms on $\mathscr{B}(\mathbb{R})$ | $\forall$dist func F, $\exists$ Borel ms μ_F on $\mathscr{B}(\mathbb{R})$ st $F=F_{\mu_F}$.
$X:(\Omega,\mathcal{F},\mathbb{P})\to(E,\mathcal{E})$: $\Omega\to_X\mathbb{R}$; $[0,1]\to_{\mathbb{P}}\mathbb{P}\to_{X^{-1}}\mathcal{B}$, / $[0,1]\to_{\mathbb{P}}\sigma(X)\to_{X^{-1}}\mathcal{B}$ (preimage)

Dist of RV

law/dist/push→/image $\mu_X:\mathcal{B}\to[0,1]:=\mathbb{P}\circ X^{-1}$. | **dist func** : $F_X(a):=\mathbb{P}(X\leq a)=\mu_X((-\infty,a])$
$X\sim Y\iff\mu_X=\mu_Y$ | **prod ms**: uniq, $\mathbb{P}(\times_iE_i)=\prod_i\mathbb{P}_i(E_i)$, $E_i\in\mathcal{F}_i$. [meas rect, $E_{\omega_1}=\{\omega_2:(\omega_1,\omega_2)\in E\}$]
<u>collection of σ-algs</u> $\mathcal{G}_i:i\in\mathcal{I}$ iff $\forall A_i\in\mathcal{G}_i$ $\mathbb{P}(\bigcap_{i\in\mathcal{I}}A_i)=\prod_{i\in\mathcal{I}}\mathbb{P}(A_i)$ for $\mathcal{I}$ fin/cntbl

Independence

If $\mathcal{G}_i:=\sigma(\mathcal{A}_i)$, <u>$\mathcal{A}_i$</u> π-sys, $(\mathcal{G}_i)_{i\in\mathcal{I}}$ indep $\iff\mathbb{P}(\bigcap_{i\in J}A_i)=\prod_{i\in J}\mathbb{P}(A_i)\forall A_i\in\mathcal{A}_i, i\in$ finite$J\subseteq\mathcal{I}$.
events $\iff$ gen. σ-algs $\iff$ part A | <u>RVs</u> $\iff(\sigma(X_i))_i\iff\mathbb{P}(X_i\in A_i,i\in J)=\prod_J\mathbb{P}(X_i\in A_i)$, finite $J\subseteq\mathcal{I}$
$\iff\mathbb{P}(X_1\leq x_1,...,X_n\leq x_n)=\mathbb{P}(X_1\leq x_1)\cdots\mathbb{P}(X_n\leq x_n)$ | <u>finite coll RVs</u> $\iff$ joint dist is prod meas of μ_{X_i} | $(X_i)_{i\in\mathcal{I}}$ indep, meas $f_i:E_i\to\mathbb{R}\Rightarrow(f_i(X_i))_{i\in\mathcal{I}}$ indep RVs.

Limits

tail σ-alg of RVs $(X_n)_{n\geq1}$ $\mathcal{T}:=\bigcap_{n=1}^\infty\mathcal{T}_n$, $\mathcal{T}_n:=\sigma(X_{n+1},X_{n+2}..)$ [$\mathcal{T}_n,\mathcal{F}_n$ ind $\Rightarrow\mathcal{T},\mathcal{F}_\infty$, $\mathbb{P}(A)=\mathbb{P}(A\cap A)$]
K's $0-1$ law: tail σ-alg of <u>indep</u> RVs: all events have prob $=0/1,\Rightarrow\mathcal{T}$-ms RV is a.s. const [$=\mathbb{P}(A)^2$]
$\{A_n$ i.o $=$ inf. often.$\}=:\limsup_{n\to\infty}A_n:=\bigcap_{n=1}^\infty\bigcup_{m\geq n}A_m=\{\omega\in\Omega:\omega\in A_m$ for ∞ many $m\}$
$\liminf A_n:=\bigcup_{n=1}^\infty\bigcap_{m\geq n}A_m=\{\omega\in\Omega:\exists m_\omega$ st $\omega\in A_m\forall m\geq m_\omega\}=\{A_n$ eventually$\}=\{A_n^C$ i.o.$\}^C$
$\mathbf{1}_{\limsup_{n\to\infty}A_n}=\limsup_{n\to\infty}\mathbf{1}_{A_n}$, same $\liminf$. | **Fatou's (+rev):** $\mathbb{P}(\liminf_{n\to\infty}A_n)\leq\liminf_{n\to\infty}\mathbb{P}(A_n)$
$\mathbb{P}(A_n\text{i.o.})=\mathbb{P}(\limsup_{n\to\infty}A_n)\geq\limsup_{n\to\infty}\mathbb{P}(A_n)$[evntl: $\mathbb{P}$ cont,^{C}] **B-C 1** $\sum\mathbb{P}(A_n)<\infty\Rightarrow\mathbb{P}(A_n$ i.o.$)=0$
[$G_n=\bigcup_{\geq n}A_m,\mathbb{P}(G_n)\leq\sum_{m=n}^\infty\mathbb{P}(A_m)\to0$ & $G_n\downarrow\limsup A_n$] **2:** A_n <u>indp</u>, $\sum_{\geq1}\mathbb{P}(A_n)=\infty\Rightarrow\mathbb{P}(A_n$ io$)=1$
[$a_m=\mathbb{P}(A_m),1-a\leq e^{-a}$, ind: $\mathbb{P}(\cap_{\geq n}A_m^c)\leq\exp(-\sum a_m)=0$ as $\sum=\infty$]

Integration & Expectation

$\int f\,\mathrm{d}\mu\equiv\int_\Omega f\,\mathrm{d}\mu\equiv\int f(\boldsymbol{\omega})\,\mu(\mathrm{d}\boldsymbol{\omega})$
Radon-Nikodym: μ,ν on $(\Omega,\mathcal{F})$, $\nu\ll\mu\iff\exists$ RV $f\geq0$ st $\nu(A)=\int_Af\,\mathrm{d}\mu$ for all $A\in\mathcal{F}$. $\mathrm{d}\nu/\mathrm{d}\mu:=f$, $v\sim\mu\iff f>0\ \mu-a.s.$, $\Rightarrow\mathrm{d}\mu/\mathrm{d}\nu=1/f$ | g is μ_X-int. $\iff g\circ X$ is $\mathbb{P}$-int, $\int_Eg(x)\,\mu_X(\mathrm{d}x)=\int_\Omega g(X(\boldsymbol{\omega}))\,\mathbb{P}(\mathrm{d}\mu)$
Fub/Ton: on prod ms spc, $f(x,y)$ ms $(\Omega,\mathcal{F})$: f is $\mathbb{P}$-int on Ω, or $\boldsymbol{f\geq0}\Rightarrow x\mapsto\int_{\Omega_2}f(x,y)\,\mathbb{P}_2(\mathrm{d}y)$ is $\mathcal{F}_1$-ms (+ v.v.), $\int_\Omega f\,\mathrm{d}\mathbb{P}=\int_{\Omega_2}\int_{\Omega_1}$ (poss. ∞) [Mon Cls] | X,Y indep $\iff\forall f,g$ ms, $f,g\geq0$ $\mathbb{E}[f(X)g(y)]=\mathbb{E}[f(X)]\mathbb{E}[g(Y)]$ [$\Rightarrow$: Fub, $\Longleftarrow:f=1_{(-\infty,r)},g=1_{(-\infty,s)}$]
a.s. $\mathbb{P}[X_n\to X]=\mathbb{P}(\{\omega:X_n(\omega)\to X(\omega)\})=1$ **in prob** $\forall\epsilon>0\ \lim_{n\to\infty}\mathbb{P}(|X_n-X|>\epsilon)=0$ a.s.$\Rightarrow$in prob
in $\mathcal{L}^p/L^p/$ pth moment $X\in\mathcal{L}^p,\forall n\,X_n\in\mathcal{L}^p$ and $\lim_{n\to\infty}\mathbb{E}[|X_n-X|^p]=0$ $\Rightarrow$in dist
weakly in $\mathcal{L}^1$ if $X_n,X\in\mathcal{L}^1$ and $\forall$bounded RVs Y $\lim_{n\to\infty}\mathbb{E}[X_nY]=\mathbb{E}[XY]$ in $L^p\Rightarrow$in prob
weakly/in dist if $\lim_{n\to\infty}F_{X_n}(x)=F_X(x)$ $\forall x\in\mathbb{R}$ at which F_X is cont.
Cheby: $X:\Omega\to\mathbb{R},\phi:\text{Im}(X)\to[0,\infty]$ incr,ms $\Rightarrow\forall\lambda\in A$ w/ $\phi(\lambda)<\infty\,\mathbb{P}[X\geq\lambda]\leq\mathbb{E}[\phi(X)]/\phi(\lambda)$ $\downarrow$ **WLLN**: ϕ's: x^2 on $|X-\mathbb{E}[X]|$, $e^{\theta x}$ gives $\mathbb{P}[X\geq\lambda]\leq e^{-\lambda\theta}\mathbb{E}[e^{\theta X}]$ | $(X_n)_{n\geq1}IID$, var $\sigma^2<\infty$, $\frac{1}{n}\sum_1^nX_i\to\mu$ in prob
Jensen's: X st $\text{Im}(X)\subseteq$interval I & $f:I\to\mathbb{R}$ convex $\Rightarrow\mathbb{E}[f(X)]\geq f(\mathbb{E}[X])$ [$\exists a:f(x)\geq f(m)+a(x-m)$]
$\mathcal{C}$ is **UI** if $\lim_{K\to\infty}\sup_{X\in\mathcal{C}}\mathbb{E}[|X|\mathbf{1}_{\{|X|>K\}}]=0$ (a.s) $\iff\sup_{X\in\mathcal{C}}\mathbb{E}[|X|]<\infty$ *AND* $\downarrow$ [HELP!!]
$\sup_{A\in\mathcal{F}:\mathbb{P}(A)\leq\delta}\sup_{X\in\mathcal{C}}\mathbb{E}[|X|\mathbf{1}_A]\to0$ as $\delta\to0$ (ignore null sets) i) $\{X\}$ is UI $\iff X$ is integ [DCT,$f_n=|X|1_{|X|>n}$];
ii) $\forall X\in\mathcal{C}$ $|X|\leq Y$ for $Y\in\mathcal{L}^1\Rightarrow\mathcal{C}$ is UI. iii) can use $(|X|-K)^+/$ similar instead iv) bdd $\Rightarrow$ UI.
Vitali's Conv: $X_n\to X$ in prob, $X_n\in\mathcal{L}^1$, tfae: 1) $\{X_n:n\geq1\}$ is UI 2) $X\in\mathcal{L}^1$ and $\mathbb{E}[|X_n-X|]\to0$ 3)$X\in\mathcal{L}^1$ and $\mathbb{E}[|X_n|]\to\mathbb{E}[|X|]<\infty$ Thus, $X_n\to X$ in $L^1\iff X_n\to X$ in prob, $\{X_n:n\geq1\}$ is UI.
Scheffé: $f_n,f\in\mathscr{L}^1(\Omega,\mathcal{F},\mu)$, $f_n\to f$ pointwise: $\int|f_n-f|\,\mathrm{d}\mu\to0\iff\int|f_n|\,\mathrm{d}\mu\to\int|f|\,\mathrm{d}\mu$. [no prf]
Conditional expectation [whole chapter is on $(\Omega,\mathcal{F},\mathbb{P})$]
$\mathbb{E}[X|\mathcal{G}]:=Y$ where Y is <u>integ.</u>, <u>$\mathcal{G}$-ms</u> & $\forall G\in\mathcal{G}$, $\mathbb{E}[Y\mathbf{1}_G]=\mathbb{E}[X\mathbf{1}_G]\iff\int_G\mathbb{E}[X|\mathcal{G}]\,d\mathbb{P}=\int_GX\,d\mathbb{P}$

Y exists, is unique a.s., can check **defining relation** only on π-sys generating $\mathcal{G}$ - e.g. open sets/atoms
for events B_n: $\mathbb{E}[X|\sigma(B_n:n\geq 1)](\boldsymbol{\omega})=\sum_{n\geq 1}\mathbb{E}[X\mathbf{1}_{B_n}]/\mathbb{P}(B_n)\mathbf{1}_{B_n}(\boldsymbol{\omega})$ (for just 1 B, use seq B, B^C)
[ALL a.s.] **a)** $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]]=\mathbb{E}[X]$ (take $G=\Omega\in\mathcal{G}$) **b)** $\mathbb{E}[X|\mathcal{G}]=X$ if X is $\mathcal{G}$-ms **c)** $\mathbb{E}[X|\{\emptyset,\Omega\}]=\mathbb{E}[X]$
d) linear **e)** $\mathbb{E}[c|\mathcal{G}]=c$ **f)** $X\leq Y\Rightarrow\mathbb{E}[X|\mathcal{G}]\leq\mathbb{E}[Y|\mathcal{G}]\Rightarrow|\mathbb{E}[X|\mathcal{G}]|\leq\mathbb{E}[|X||\mathcal{G}]$
g)$\mathbb{E}[X|\mathcal{G}]=\mathbb{E}[X]$ if $\sigma(X),\mathcal{G}$ indep **h)** $\mathbb{E}[X|Z]:=\mathbb{E}[X|\sigma(Z)]$ is $\sigma(Z)$ ms, i.e. a function of Z.
cMCT: $X_n\geq 0, X_n\uparrow X\Rightarrow\mathbb{E}[X_n|\mathcal{G}]\uparrow\mathbb{E}[X|\mathcal{G}]$ a.s.; **cFatou**: $X_n\geq 0\Rightarrow\mathbb{E}[\liminf_{n\to\infty}X_n|\mathcal{G}]\leq\liminf_{n\to\infty}\mathbb{E}[X_n|\mathcal{G}]$ a.s.;
cDCT: Y integrable, $|X_n|\leq Y, X_n\to X$ a.s$\Rightarrow\mathbb{E}[X_n|\mathcal{G}]\to\mathbb{E}[X|\mathcal{G}]$ a.s.
take out known: X,Y rvs with X,Y,XY integ, Y $\mathcal{G}$-ms. Then $\mathbb{E}[XY|\mathcal{G}]=Y\cdot\mathbb{E}[X|\mathcal{G}]$ a.s.
tower property: $X\in\mathcal{L}^1,\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\mathcal{F}$ then $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1]=\mathbb{E}[X|\mathcal{F}_1]$ a.s.
cJensen: $X\in\mathcal{L}^1,\mathrm{Im}(X)\subseteq\text{intvl } I$, $f:I\to\mathbb{R}$ convex, $\mathbb{E}[|f(X)|]<\infty\Rightarrow\mathbb{E}[f(X)|\mathcal{G}]\geq f(\mathbb{E}[X|\mathcal{G}])$ a.s.
X an integ. RV, $\{\mathcal{F}_\alpha:\alpha\in I\}$ σ-algs st $\forall\alpha\,\mathcal{F}_\alpha\subseteq\mathcal{F}$ then $\{X_\alpha:=\mathbb{E}[X|\mathcal{F}_\alpha]:\alpha\in I\}$ is UI.
Total $\mathbb{E}$ (for calc. only) $\mathbb{E}[X]=\sum_i\mathbb{E}[X\mathbf{1}_{A_i}]=\sum_i\int X\,\mathbb{P}(d\boldsymbol{\omega}\,|\,A_i)\cdot\mathbb{P}(A_i)\approx\sum_i\mathbb{E}[X\,|\,A_i]\mathbb{P}[A_i]$
covariance $\mathrm{Cov}(X,Y):=\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]=\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]$; **uncorrelated**: $\mathrm{Cov}(X,Y)=0$
$\langle X,Y\rangle:=\mathbb{E}[XY]$ is a scalar prod for $X,Y\in\mathscr{L}^2$ - use for proof cond exp. exists by orthog., closest point

Filtration & stopping times

A **filtration** on $(\Omega,\mathcal{F},\mathbb{P})$: a seq $(\mathcal{F}_n)_{n\geq 0}$ of σ-algs $\mathcal{F}_n\subseteq\mathcal{F}$ st $\mathcal{F}_n\subseteq\mathcal{F}_{n+1}$ $\forall n$. gens. $\mathcal{F}_\infty:=\sigma(\bigcup\mathcal{F}_n)$
(X_n) **adapted** to $(\mathcal{F}_n)$: if $\forall n$ X_n is $\mathcal{F}_n$-ms, **integrable** if $\forall n\,X_n$ is. natural filt: $\mathcal{F}_n^X=\sigma(X_0,X_1,...,X_n)$
stopping time wrt $(\mathcal{F}_n)_{n\geq 0}$: RV $\tau:\Omega\to\mathbb{N}\cup\{\infty\}$ st $\forall n$ $\{\tau\leq/\geq/</>n\}\in\mathcal{F}_n$. - e.g. const, min, max,
hitting time $h_B:=\inf\{n\geq 0:X_n\in B\}$ for $(X_n)_{n\geq 0}$ adapted, B Borel.
The σ-alg at stopping time τ: $\mathcal{F}_\tau:=\{A\in\mathcal{F}_\infty:\forall n\geq 0\,A\cap\{\tau=/\leq/\geq/</>n\}\in\mathcal{F}_n\}$.
$\tau\leq\rho\Rightarrow\mathcal{F}_\tau\subseteq\mathcal{F}_\rho$; X_τ is an RV if $\tau<\infty$, defined as $\omega\mapsto(X_{\tau(\omega)})(\omega)$, and is $\mathcal{F}_\infty$ and $\mathcal{F}_\tau$-ms.
stopped process of $(X_n)_{n\geq 0}$ and τ: $X^\tau=(X_{\tau\wedge n})_{n\geq 0}$, which is adapted to $(\mathcal{F}_{\tau\wedge n})_{n\geq 0}$ and $(\mathcal{F}_n)_{n\geq 0}$

Martingales in discrete time

$(X_n)_{n\geq 0}$, integ, $\mathcal{F}_n$-adapted is a **martingale** if $\forall n\geq 0\,\mathbb{E}[X_{n+1}|\mathcal{F}_n]=X_n$ a.s., **sub/super**mart if $\geq/\leq$
$\mathbb{E}[X_n|\mathcal{F}_m]=/\geq/\leq X_m$ a.s. if $n\geq m$ $\mathbb{E}[X_n]=/\geq/\leq\mathbb{E}[X_m]\cdots\mathbb{E}[X_0]$ a.s. if $n\geq m$
mart diff seq wrt $(\mathcal{F}_n)$: integ. $(Y_n)_{n\geq 1}$ st $\forall n\geq 0\,\mathbb{E}[Y_{n+1}\,|\,\mathcal{F}_n]=0$ a.s.
$(f(X_n))_{n\geq 0}$ is a submart if f convex on $\mathbb{R}$, $(X_n)_{n\geq 0}$ a (sub)mart. E.g.: $|X_n|$, $X_n^2, e^{X_n}e^{-X_n},\max\{X_n,K\}$
$(V_n)_{n\geq\mathbf{1}}$ is **predictable** on $(\mathcal{F}_n)_{n\geq 0}$ if $\forall n\geq 1$, V_n is $\mathcal{F}_{n-1}$-ms
mart transform of pred. $(V_n)_{n\geq\mathbf{1}}$, mart $(X_n)_{n\geq 1}$: $((V\circ X)_n)_{n\geq 0}:=(\sum_{k=1}^n V_k(X_k-Y_{k-1}))_{n\geq 0}$
Doob's Decomp. integ. adapted $\boldsymbol{X}=(X_n)_{n\geq 0}$ decomposes $X_n=X_0+M_n+A_n$ for M mart, A pred.
uniq in prob: $\mathbb{P}(M_n=M_n', A_n=A_n'\,\forall n\geq 0)=1$; $\boldsymbol{X}$ is subm $\iff$ $(A_n)_{n\geq 0}$ incr a.s., mart = const a.s.
$\langle M\rangle_n=A_n$ for a L^2-mart $\boldsymbol{M}$ ($\mathbb{E}[M_n^2]<\infty$), Doob decomp $M_n^2=M_0^2+N_n+A_n$. A_n incr as x^2 convex
$\boldsymbol{X}^\tau:=(X_{\tau\wedge n}=X_{n\wedge\tau(\boldsymbol{\omega})}(\boldsymbol{\omega}))_{n\geq 0}$: **stopped process** of $\boldsymbol{X}$, finite τ, is a mart wrt $(\mathcal{F}_{\tau\wedge n})_{n\geq 0}$, $(\mathcal{F}_n)_{n\geq 0}$
Doob Opt Samp/Stop: bdd $\tau\leq\rho$: $\mathbb{E}[X_\rho]=\mathbb{E}[X_\tau]=\mathbb{E}[X_0]$ & $\mathbb{E}[X_\rho|\mathcal{F}_\tau]=X_\tau$ a.s. (same for sub/sup)
OST variant : τ *a.s.* finite $\Rightarrow\mathbb{E}[X_\tau]=\mathbb{E}[X_\tau\mathbf{1}_{\tau<\infty}]=\mathbb{E}[X_0]$ if either: $\{X_n:n\geq 0\}$ is UI
or $\mathbb{E}[\tau]<\infty$ and $\exists L\in\mathbb{R}$ st $\forall n\,\mathbb{E}[|M_{n+1}-M_n||\mathcal{F}_n]\leq L$ a.s. **Doob maximal ineq**: $(X_n)_{n\geq 0}$ submart,
$\forall\lambda>0$, $Y_n^\lambda:=(X_n-\lambda)\mathbf{1}_{\{\max_{k\leq n}X_k\geq\lambda\}}$ submt & $\lambda\mathbb{P}[\max_{k\leq n}X_n\geq\lambda]\leq\mathbb{E}[X_n\mathbf{1}_{\{\max_{k\leq n}X_k\geq\lambda\}}]\leq\mathbb{E}[|X_n|]$.
for $p\geq 1,(M_n)_{n\geq 0}$ mart $M_n\in\mathscr{L}^p$, $\forall N\geq 0,\lambda>0$ $\mathbb{P}[\max_{n\leq N}|M_n|\geq\lambda]\leq\mathbb{E}[|M_N|^p]/\lambda^p$
Db L^p ineq: $p>1,(X_n)\in\mathscr{L}^p$ non-neg subm, $\overline{X}_n:=\max_{\leq n}X_k\in\mathscr{L}^p$, $\mathbb{E}[X_n^p]\leq\mathbb{E}[\max_{k\leq n}X_k^p]\leq(\frac{p}{p-1})^p\mathbb{E}[X_n^p]$
— (X_n) supermart: $\forall\lambda,n\geq 0$ then $\lambda\mathbb{P}(\max_{k\leq n}|X_k|\geq\lambda)\leq\mathbb{E}[X_0]+2\mathbb{E}[X_0]+2\mathbb{E}[X_n^-]$
$\boldsymbol{x}=(x_n)\in\mathbb{R}^\mathbb{N}$, fix $a<b$: $(\rho_k)_{k\geq\mathbf{1}},(\tau_k)_{k\geq\mathbf{0}}$ $\tau_0=0$ $\rho_k=\inf\{n\geq\tau_{k-1}:x_n\leq a\}$ $\tau_k=\inf\{n\geq\rho_k:x_n\geq b\}$
 # upcrossings $U_n([a,b],\boldsymbol{x})=\max\{k:\tau_k\leq n\}$ **total** $U([a,b],\boldsymbol{x})=\sup_n U_n([a,b],\boldsymbol{x})=\sup\{k:\tau_k<\infty\}$
DB upcrossings: $\boldsymbol{X}=(X_n)_{n\geq 0}$ supermart, $a<b$ fixed, $\forall n\geq 0$: $\mathbb{E}[(U_n([a,b],\mathbf{X})]\leq\mathbb{E}[(X_n-a)^-]/(b-a)$
$\boldsymbol{x}\in\mathbb{R}^\mathbb{N}$ conv $\iff\forall a,b\in\mathbb{Q},a<b\,U([a,b],\boldsymbol{x})<\infty$ — $(X_n)_{n\geq 1}$ is **bdd in** L^p if $\sup_n\mathbb{E}[|X_n|^p]<\infty$
DB conv $\boldsymbol{X}$ sub/super, bdd in L^1, $\boldsymbol{X}\to X_\infty:=\liminf_{n\to\infty}X_n$ **a.s.**, X_∞ integrable
$(X_n)_{n\geq 0}$ a non-neg supermart, $X_\infty=\lim_{n\to\infty}X_n$ exists a.s. (no need for L^1, as $\mathbb{E}[|X_n|]=\mathbb{E}[X_n]\leq\mathbb{E}[X_0]$)
TFAE for mart: a) is UI b) $\exists\mathcal{F}_\infty$-ms $M_\infty:M_n\to M_\infty$ in L^1+a.s. c) $\exists\mathcal{F}_\infty$-ms $M_\infty:\forall n\,M_n=\mathbb{E}[M_\infty\,|\,\mathcal{F}_n]$
$M_\infty\in\mathscr{L}^p$, $p>1\Rightarrow M_n\to M_\infty$ in $\mathscr{L}^p$ — $\boldsymbol{M}$ UI mart, $\forall\tau\leq\rho$ $\mathbb{E}[M_\rho\,|\,\mathcal{F}_\tau]=M_\tau$ a.s. & $\mathbb{E}[M_\tau]=\mathbb{E}[M_0]$
$\boldsymbol{M}$ UI mart: $M_\infty^*:=\max_{n\geq 0}|M_n|$, $\lambda\mathbb{P}[M_\infty^*\geq\lambda]\leq\mathbb{E}[|M_\infty|\mathbf{1}_{\{M_\infty^*\geq\lambda\}}]$ for $\lambda\geq 0$.
$M_\infty\in\mathscr{L}^p$, $p>1$, $1/p+1/q=1$: $\|M_\infty\|_p\leq\|M_\infty^*\|_p\leq q\|M_\infty\|_p$, & $M_n\to M_\infty$ in $\mathscr{L}^p$.
Backwards: time in $I=\{t\in\mathbb{Z}:t\leq 0\}$ ends at 0. filts: $(\mathcal{F}_{-n})_{n\geq 0}$, with $\forall n:\mathcal{F}_{-n}\subseteq\mathcal{F}$ & $\mathcal{F}_{-k}\subseteq\mathcal{F}_{-k+1}$
M_{-n} is a **backwards mart** if $\forall n:M_{-n}$ integ. and $\mathcal{F}_{-n}$ ms, and $\mathbb{E}[M_{-n+1}\,|\,\mathcal{F}_{-n}]=M_{-n}$ a.s.
$M_{-n}=\mathbb{E}[M_0\,|\,\mathcal{F}_{-n}]$ a.s., so $(M_{-n})_{n\geq 0}$ is UI [tower - CHECK]. Doob's Upcrossing holds, $M_{-n}\to M_{-\infty}=$
$\mathbb{E}[M_{-1}|\mathcal{F}_{-\infty}]$, conv is a.s. & in L^1. $\mathcal{F}_{-\infty}=\bigcap_{k=0}^\infty\mathcal{F}_{-k}$ - as $k\uparrow$, $\mathcal{F}_{-k}$ decr. - $M_{-\infty}$ is $\mathcal{F}_{-\infty}$,$\mathcal{F}_{-k}$ ms
Kolmogorov's SLLN: IID, integ. w/ mean m $(X_n)_{n\geq 1}$: $S_n=\sum_{k=1}^n X_k\to m$ a.s. and in L^1 as $n\to\infty$.