algebra: $\mathcal{F} \subseteq P(\Omega)$, $\emptyset \in \mathcal{F}$, $E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}$, $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$ σ-algebra: $\Leftarrow + E_n \in \mathcal{F} \Rightarrow \bigcup_n E_n \in \mathcal{F} \Rightarrow \bigcap E_n \in \mathcal{F}$ | countable $\bigcap$ of σ-alg = σ-alg.
prod space: of $(\Omega_i, \mathcal{F}_i)_{i \in I}$: cartesian prod $\prod_i \Omega_i$, $\times \mathcal{F}_i := \sigma($cylinder sets $= \prod_i A_i, A_i \in \mathcal{F}_i\ \forall i$ except fin $A_i = \Omega_i)$ | Borel σ-a: gen by open sets
π-sys: $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$ | λ-sys: $\Omega \in \mathcal{A}$, $A, B \in \mathcal{A} \Rightarrow B \setminus A \in \mathcal{A}$, $A_n \in \mathcal{A}$, $A_n \subseteq A_{n+1} \Rightarrow \bigcup_n A_n \in \mathcal{A}$ | σ-alg $\Leftrightarrow$ π & λ $\Leftarrow B_n := \bigcup_k^n A_k$, B_n fin so π]
π-λ: $\mathcal{M} =$ λ-s, $\mathcal{A} =$ π-s, $\mathcal{A} \subseteq \mathcal{M} \Rightarrow \sigma(\mathcal{A}) \subseteq \mathcal{M}$ $\lambda(\mathcal{A}) := \bigcap$ λ-s cont $\mathcal{A}$, $\lambda(\mathcal{A}) \subseteq \mathcal{M}$. want $\lambda(\mathcal{A}) =$ σ-a. $\{A \in \lambda(\mathcal{A}) : A \cap C \in \lambda(\mathcal{A})\ \forall C \in \mathcal{A}\}$ then $\{A \in \lambda(\mathcal{A}) : A \cap C \in \lambda(\mathcal{A})\ \forall C \in \lambda(\mathcal{A})\}$.
$\sigma(X) = \{X^{-1}(E) : E \in \mathcal{E}\} = \sigma(X^{-1}(A) : A \in \mathcal{A})$, $\sigma(\mathcal{A}) = \mathcal{E}$, $X : (\Omega, \mathcal{F}) \to (\mathbb{R}', \mathcal{E})$ | RV g on $(\Omega, \mathcal{E})$ is $\sigma(X)$-meas $\Leftrightarrow g = h \circ X$, h an RV on $(\Omega', \mathcal{E})$ no pf | bdd $\sigma(X)$-ms
Monotone class theorem: $\mathcal{H}$ is a class of bdd funcs $\Omega \mapsto \mathbb{R}$, vect space over $\mathbb{R}$, $1 \in \mathcal{H}$, $f_n \in \mathcal{H}$, $f_n \uparrow f$ bdd $\Rightarrow f \in \mathcal{H}$ $\mathcal{C} \subseteq \mathcal{H}$ closed under pwise $\times \Rightarrow$ funcs $\in \mathcal{H}$
$\mu : \mathcal{F} \to [0, \infty]$ a measure on $(\Omega, \mathcal{F})$: 1) $\mu(\emptyset) = 0$ 2) $\mu(\bigcup_n^\infty E_n) = \sum_n^\infty \mu(E_n)$ if disjoint. props 1) $A \cap B = \emptyset \Rightarrow \mu(A + B) = \mu(A) + \mu(B)$ 2) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$
3) $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ 4) $A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$ 5) $B_n \downarrow B$, $\exists k : \mu(B_k) < \infty \Rightarrow \mu(B_n) \downarrow \mu(B)$ 6) $\mu(\bigcup_{n \geq 1} A_n) \leq \sum_{n \geq 1} \mu(A_n)$
σ-finite $\exists K_n \in \mathcal{F}$ st $\forall n$ $\mu(K_n) < \infty$, $\bigcup_{n \geq 1} K_n = \Omega$ abs cont & equiv $\nu \ll \mu$: $\forall A$ $\mu(A) = 0 \Leftrightarrow \nu(A) = 0$ | $\forall E$ $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$
Uniq ext: $\mathcal{F} = \sigma(\mathcal{A})$, $\mathcal{A} =$ π-sys $\mu_1 = \mu_2$ on $\mathcal{A}$, $\mu_1(\Omega) = \mu_2(\Omega) < \infty \Rightarrow \mu_1 = \mu_2$ on $\mathcal{F}$ $\{A : \mu_1 = \mu_2\}$ is a λ-s | outer meas $\mu^*(B) = \inf\{\sum^\infty \mu_0(A_i) : A_i \in \mathcal{A}, \bigcup^\infty A_i \supseteq B\}$
Carathéodory: $\mathcal{F} = \sigma(\mathcal{A})$, $\mathcal{A} =$ an alg $\mu_0 : \mathcal{A} \to [0, \infty]$ countably additive & $\mu(\emptyset) = 0 \Rightarrow$ outer meas ext. μ_0 to $\mathcal{F}$. Distributions
dist func: $F : \mathbb{R} \to [0, 1]$ | $F_\mu(x) := \mu((-\infty, x]) =$ dist func. | F a dist func $\Rightarrow \exists$ Borel meas | Law / dist. / pushforward / image measure of X: $\mu_X : \mathcal{B} \to [0, 1]$
a) F increasing | Lebesgue meas $\exists$: Leb_k on $(k, k+1]$, $A \cap (k..] \Leftarrow$ i.e. μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ st $F_\mu = F$ | prod measure: uniq ms st $\mu_X = \mathbb{P} \circ X^{-1}$.
b) $x \to \infty, -\infty$ $F(x) \to 1, 0$ uniq: as $\pi(\mathbb{R}) = \{(-\infty, x]\}$ is a π-sys gen $\mathcal{B}(\mathbb{R})$ so L10. exists: $I_{a,b} = (a, b]$ | $\mathbb{P}(\times_i E_i) = \prod_i \mathbb{P}[E_i]$ meas rect $= A \times B \Leftarrow$ π-sys, gen $\mathcal{F}$.
c) right cont: $y \downarrow x$ $F(y) \to F(x)$ | $\mathcal{J} := \{I_{a,b} : a \leq b \in [-\infty, \infty]\}$, $\mathcal{A} =$ finite disj unions of $\mathcal{J} =$ alg $I_{a,\infty} = (a, \infty)$ | $\mathcal{A} =$ finite unions of disj. m.rects, $\mathbb{P}(\cdot)$ on $\mathcal{A}$ is set func. is count add: $D_n \in \mathcal{A}$, $\downarrow \emptyset$
$\sigma(\mathcal{A}) = \sigma(\mathcal{J}) = \mathcal{B}(\mathbb{R})$. $\mu_0(I_{a,b}) = F(b) - F(a)$ = set func. μ_0 count add: $\Leftarrow$ to $\downarrow \emptyset$ $\mu_0(A_n) \to 0$ lemma, $(A_n)_{n \geq 1}$ | $\mathbb{P}(D_n \to 0)$: $D_n = A_{n,k} \times B_{n,k}$. WLOG A_n disjoint. $D_n(\omega) = B_{n,k} \Leftrightarrow \omega \in A_{n,k} \in \mathcal{F}_2$
want compact: $B_n = A_n \cap [-l, l]$ lemma: additive set func $< \infty$ on alg: count add $\Leftrightarrow \forall A_n \in \mathcal{A}$, $A_n \downarrow \emptyset$ $\mu(A_n) \to 0$. | so $D_n(\omega) \downarrow \emptyset$. $X_n(\omega_1) := \mathbb{P}(D_n(\omega_1))$ $X_n \downarrow \emptyset$ pwise, $X_n =$ simple func. $X_n^{-1}(\varepsilon, \infty)$
a collection $(\mathcal{G}_i)_{i \in I}$ of σ-alg is independent $\Leftrightarrow \forall A_i \in \mathcal{G}_i$ $\mathbb{P}(\bigcap_i A_i) = \prod_i \mathbb{P}(A_i)$ for I finite or countable $\mathbb{P}(D_n) \leq \mathbb{P}(X_n > \varepsilon) + \varepsilon$
events $B_i \Leftrightarrow \sigma(B_i)$ indep $\Leftrightarrow$ standard, same for RVs. $\Leftarrow\Rightarrow$ $\mathcal{G}_i = \sigma(\mathcal{A}_i)$, $\mathcal{A}_i :=$ π-sys, $\hookrightarrow$ for $A_i \in \mathcal{A}_i$, $i \in J$ finite $\subseteq I$
finite # of RVs $\Leftrightarrow$ joint dist = prod meas of μ_{X_i}. | $(X_i)_{i \in I}$ indep, $f_i : E_i \to \mathbb{R}$ meas. $\Rightarrow (f_i(X_i))_{i \in I}$ indep as $Y = f(X) \Rightarrow \sigma(Y) \subseteq \sigma(X)$.
tail σ-alg: of $(X_n)_{n \geq 1}$: $\mathcal{T}_n := \sigma(X_{n+1}, X_{n+2}, \ldots)$ $\mathcal{T} = \bigcap_{n \geq 1} \mathcal{T}_n$ | Kolmogorov 0-1: $\mathcal{T}$ of indep RVs, $\forall A \in \mathcal{T}$ $\mathbb{P}(A) = 0$ or 1
$\{A_n \text{ i.o.}\} = \limsup_n A_n = \bigcap_{n=1}^\infty \bigcup_{m \geq n} A_m = \{\omega : \omega \in A_n$ for ∞ many $n\}$ | $\Rightarrow \mathcal{T}$-meas RV is a.s. const $\mathcal{T}_n, \mathcal{F}_n$ ind $\Rightarrow \mathcal{T}, \mathcal{F}_\infty$ ind, $\mathbb{P}(A) = \mathbb{P}(A \cap A)$
$\{A_n$ eventually$\} = \{A_n^c \text{ i.o.}\}^c = \liminf_n A_n = \bigcup_{n=1}^\infty \bigcap_{m \geq n} A_m = \{\omega : \exists m_\omega\ \forall m \geq m_\omega\ \omega \in A_m\}$. | $1_{\limsup A_n} = \limsup 1_{A_n}$ $\mathcal{T} \subseteq \mathcal{F}_\infty \Rightarrow \nearrow = \mathbb{P}(A)^2$
Fatou: $\mathbb{P}(\liminf_n A_n) \leq \liminf_n \mathbb{P}(A_n)$ $\mathbb{P}(\bigcap_{m \geq n} A_m) \leq \inf_{m \geq n} \mathbb{P}(A_m)$ reverse: $\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}(\limsup A_n) \geq \limsup \mathbb{P}(A_n)$ $\leftarrow$ + same $\liminf$ complement
Borel-Cantelli 1: $\sum \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(A_n \text{ i.o.}) = 0$ $G_n = \bigcup_{m \geq n} A_m$, $\mathbb{P}(G_n) \leq \sum_{m \geq n}^\infty \mathbb{P}(A_m) \to 0$, $G_n \downarrow \limsup$ still $= \infty$
2: A_n indep $\sum \mathbb{P}(A_n) = \infty \Rightarrow \mathbb{P}(A_n \text{ i.o.}) = 1$ $a_m := \mathbb{P}(A_m)$ $1 - a \leq e^{-a}$, $\mathbb{P}(\bigcap_{m \geq n} A_m^c) = \prod_{m \geq n}(1 - a_m)$ (indep) $\leq \exp(-\sum a_m) = 0$ as tail of $\sum = \infty$
reverse Fatou $f_n \geq 0$, $f_n \leq g$ integ $\Rightarrow \int \limsup f_n\, d\mu \geq \limsup \int f_n\, d\mu$ DCT: $f_n \to f$ pwise, $|f_n| \leq g$ integ $\Rightarrow f$ integ, $\int f_n\, d\mu \to \int f\, d\mu$
Radon-Nikodym $\nu \ll \mu \Rightarrow \exists$ RV $f \geq 0$ st $\nu(A) = \int_A f\, d\mu$ $\forall A \in \mathcal{F}$, $d\nu/d\mu := f$, $\nu \sim \mu \Leftrightarrow f > 0$ a.s. $d\mu/d\nu = 1/f$ no pf. MCT: $f_n \uparrow f \Rightarrow \int f_n\, d\mu \uparrow \int f\, d\mu$
g is μ_X integ $[\int g\, d\mu_X < \infty] \Leftrightarrow g \circ X$ is $\mathbb{P}$-int, $\int_E g(x)\, \mu_X(dx) = \int_\Omega g(X(\omega))\, \mathbb{P}(d\omega)$ start $g = 1_A$ lin, MCT, $g^+ - g^-$ Fatou $\int \liminf f_n\, d\mu \leq \liminf(\int f_n\, d\mu)$
Fubini/Tonelli: $f(x,y)$ meas on prod space: $x \mapsto \int_{\Omega_2} f(x,y)\, \mathbb{P}_2(dy)$ is $\mathcal{F}_1$ meas + v.v, f $\mathbb{P}$ integ or $f \geq 0$: $\int f = \int_{\Omega_1} \int_{\Omega_2} f(x,y)\, \mathbb{P}_1(dx)\, \mathbb{P}_2(dy) =$ v.v [may be ∞ if only $f \geq 0$]
X, Y indep $\Leftrightarrow \forall f, g$ meas, $f, g \geq 0$ $\mathbb{E}[f(X) g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$ $\Rightarrow$ Fub, $\Leftarrow f = 1_{(-\infty, x]}$ pf: Mon Class: $\mathcal{H} = f$ s.t. $\mathcal{K}$, $\mathcal{C} =$ simple, $\subseteq \mathcal{H}$ as $\sigma(\mathcal{C})$

a.s. $\mathbb{P}(X_n \to X) = \mathbb{P}(\{\omega : X_n(\omega) \to X(\omega)\}) = 1$ | a.s $\Rightarrow$ in prob $\not\Leftarrow$: $[0,1]$, Borel, Leb meas: $X_n(\omega) = 1_{[k/2^m, (k+1)/2^m]}(\omega)$, $n = 2^m + k \Rightarrow$: $A_{n,\varepsilon} = \{|X_n - X| > \varepsilon\}$
in prob $\forall \varepsilon > 0$ $\lim_n \mathbb{P}(|X_n - X| > \varepsilon) = 0$ | prob $\Rightarrow$ in dist split $\mathbb{P}(X \leq a) \leq \mathbb{P}(X \leq a + \varepsilon)$, $\mathbb{P}(|X_n - X| > \varepsilon)$ a.s. $\Rightarrow \mathbb{P}(A_{n_k} \text{ i.o.}) = 0 \Rightarrow \limsup \mathbb{P}(A_n)$ Fat
in $\mathcal{L}^P$ $X \in \mathcal{L}^P$, $\forall n$, $X_n \in \mathcal{L}^P$, $\lim_n \mathbb{E}[|X - X_n|^P] = 0$ | prob $\Rightarrow$ subseq a.s. $\mathbb{P}(A_{n,1/k}) \to 0$ for each k $[A_n^{1/k}] \Rightarrow \exists n_k$ st $\mathbb{P}(\cdot) < 1/k^2$, $n_k > n_{k-1}$, $B_k = A_{n_k, 1/k}$ B-C I
weakly in $\mathcal{L}^1$ $X, X_n \in \mathcal{L}^1$, $\forall$ bdd RVs Y $\mathbb{E}[X_n Y] \to \mathbb{E}[XY]$ | $\mathcal{L}^P \Rightarrow$ prob split $\mathbb{E}[\ ]$ $\mathbb{P}[\ ]$ in $> \varepsilon, \leq \varepsilon$. so $\mathbb{P}(B_k \text{ i.o.}) = 0$, so fin many $B_k \Rightarrow$ a.s. conv
weakly / in dist $F_{X_n}(x) \to F_X(x)$ $\forall x \in \mathbb{R}$ at which F_X is cont | a.s $\not\Rightarrow$ in $\mathcal{L}^1$ $f_n(x) = \{n(1 - nx)$ $x \in [0, 1/n]$ else $0\}$ $f_n \to 0$ a.s. [a.e.], $\not\to 0$ in $\mathcal{L}^1$.
Chebyshev: $X : \Omega \to \mathbb{R}$, $\phi : [\ldots] \to [0, \infty]$ incr, meas $\Rightarrow \forall \lambda$ st $\phi(\lambda) < \infty$, $\mathbb{P}(X \geq \lambda) \leq \mathbb{E}[\phi(X)] / \phi(\lambda)$. $\mathbb{P}(X \geq \lambda) = \mathbb{P}(\phi(X) \geq \phi(\lambda)) \to$ Markov: $\mathbb{P}(X \geq x) = 1/x\, \mathbb{E}[X]$.
useful ϕ's: x^2 on $|X - \mathbb{E}[X]|$ gives standard, $e^{\theta x}$ gives $\leq e^{-\lambda\theta}\, \mathbb{E}[e^{\theta x}]$, then optimise over θ (an MGF). WLLN: X_n i.i.d, var $< \infty$, Cheb on $(S_n/n)_{n \geq 1}$
Jensen's: X, $f : I \supseteq X \to \mathbb{R}$ convex, $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$ lemma: f conv $\Rightarrow \forall x_0 \exists a$ $f(x) \geq f(x_0) + a(x - x_0)$ $1/n \sum_{i=1}^n X_i \to \mu$ in prob
UI: collection $\mathcal{C}$ of RVs is uniformly integrable if $\lim_{K \to \infty} \sup_{X \in \mathcal{C}} \mathbb{E}[|X| 1_{\{|X| > K\}}] = 0$ | $0 \leq r \leq p$: $X \in \mathcal{L}^P \Rightarrow X \in \mathcal{L}^r$ Jensen on $|X| \wedge k$ w/ $x^{p/r}$, then MCT
Holder: $p, q > 1$ $X, Y \in \mathcal{L}^P$ $\mathbb{E}[|XY|] \leq \|X\|_p \|X\|_q$ $Q(A) = \mathbb{E}[|X|^p 1_A] / \|X\|_p^p$
$\Leftrightarrow$ a) $\sup_X \mathbb{E}[|X|] < \infty$ $\Leftrightarrow \forall \varepsilon > 0\ \exists K$ st $\forall X \in \mathcal{C}$ $\mathbb{E}[|X| 1_{\{|X| > K\}}] < \varepsilon$ | Minkowski: $1/p + 1/q = 1$ $Z \in \mathcal{L}^1$ $\|X + Y\|_p \leq \|X\|_p + \|Y\|_p$ $Z = (|X|/\ldots)$
& b) $\sup_{A \in \mathcal{F} : \mathbb{P}(A) \leq \delta} \sup_X \mathbb{E}[|X| 1_A] \to 0$ as $\delta \to 0$ | $x\mathbb{P}(X \geq x) \leq \mathbb{E}[Y 1_{X \geq x}]$, $X, Y \geq 0 \Rightarrow \|X\|_p \leq q \|Y\|_p$ (if $1/p + 1/q = 1$)
$\Rightarrow$ (UI: so $\exists K$ $\mathbb{E}[|X| 1_{|X| > K}] \leq 1$ $\forall X$. $\Rightarrow$ a) by splitting $\leq K, > K$. $\forall \varepsilon > 0$ K st $\mathbb{E}[\ldots] < \frac{1}{2}\varepsilon$ | Fubini to get $\mathbb{E}[X^p] \leq q\, \mathbb{E}[X^{p-1} Y]$, Holder assuming $X \in \mathcal{L}^P$, general: $X_n \wedge n$ & MCT
$\delta = \varepsilon / 2K$, $\mathbb{P}(A) \leq \delta$: $\mathbb{E}[|X| 1_A] =$ split on $\lessgtr K \leq \frac{1}{2}\varepsilon + K\, \mathbb{P}(A) < \varepsilon$ | $\mathcal{L}^P$ is complete for $p \geq 1$. $\mathcal{L}^P$ spaces
$\Leftarrow$ $\forall \varepsilon > 0$, take δ from b). $M := \sup \mathbb{E}[|X|]$ $K > M/\delta$: $\mathbb{P}(|X| > K) \leq M/K < \delta$ by Markov | 2) $\forall X$ $|X| < Y$ in $\mathcal{L}^1 \Rightarrow \{X\}$ UI by DCT, same idea as 1)
notes about UI: 1) $\{X\}$ UI $\Leftrightarrow X$ integ [DCT on $f_n = |X| 1_{|X| > n}$, $\Rightarrow \mathbb{E}$ decr in K] | 3) can use $(|X| - K)^+$ instead or instead of $|X| 1_{|X| > K}$ 4) $\forall X$ $|X| \leq c \Rightarrow \{X\}$ UI
$X_n \to X$ in prob, $|X_n|, |X| \leq K \in \mathbb{R} \Rightarrow X_n \to X$ in $\mathcal{L}^1$ $\hookrightarrow$ extends to finite set | 5) bounded in $\mathcal{L}^P$, $p > 1 \Rightarrow$ UI $|X_i|^k \leq L$. $|X_i| > x \Rightarrow |X_i|^k = |X_i||X_i|^{k-1} \leq |X_i| x^{k-1}$
$\mathbb{E}[|X_n - X|]$, split on $A_n = |X_n - X| > \varepsilon$, $\leq \mathbb{E}[|X| 1_{A_n}] + \mathbb{E}[|X_n| 1_{A_n}] + \varepsilon \leq 2K\mathbb{P}(A_n) + \varepsilon$. | 6) closed under linear combinations | $\Rightarrow |X_i| \leq |X_i|^k / x^{k-1}$
Vitali: $X_n \to X$ in prob TFAE: 1) $\{X_n\}$ UI $\to 0$ $\uparrow$ as $X_n \to X$ in pr. | 7) $X_n \in \mathcal{L}^1$, $\mathbb{E}[|X - X_n|] \to 0 \Rightarrow \{X_n\}$ UI $\mathbb{E}[|X_n| 1_{|X_n| > K}] \leq \mathbb{E}[(|X_n - X| + |X|) 1_{|X_n| > K}]$
$1 \Rightarrow 2)$ as on L50, $X \in \mathcal{L}^1$: subseq $X_{n_k} \to X$ a.s 2) $X \in \mathcal{L}^1$, $\mathbb{E}[|X_n - X|] \to 0$ as $n \to \infty$ | [ε: if $n > N$ then conv $\Rightarrow \mathbb{E}[X_n - X] < \varepsilon$, else $K \gg 0$ st $\mathbb{P}(|X_n| > K) < \delta_n \Leftarrow X_n - X$ UI] $\leq \varepsilon + \mathbb{E}[|X|]\, \mathbb{P}(|X_n| > K)$
Fatou: $\mathbb{E}[|X|] \leq \liminf \mathbb{E}[|X_n|] \leq \sup_n \mathbb{E}[|X_n|]$ 3) $X \in \mathcal{L}^1$ $\mathbb{E}[|X_n|] \to \mathbb{E}[|X|] < \infty$ | $\mathbb{E}[(Y_n - K)^+] = \mathbb{E}[Y_n] - \mathbb{E}[Y_n \wedge K] \to \mathbb{E}[(Y - K)^+] < \varepsilon$ for large K.
$2 \Rightarrow 3$: $-|X_n - X| \leq |X| - |X_n| \leq |X_n - X|$ $3 \Rightarrow 1)$ $Y = |X|$, $Y_n = |X_n|$ so $Y_n \xrightarrow{P} Y$ $Y_n \wedge K \xrightarrow{P} Y \wedge K$, use L49. | so $\exists N$ st $n \geq N \Rightarrow \mathbb{E}((|X_n| - K)^+) < 2\varepsilon$, fin. many $n < N$, so $\exists K' \geq K$ st ✓ $\forall n$.
so $X_n \to X$ in $\mathcal{L}^1 \Leftrightarrow X_n \to X$ in prob & $\{X_n\}$ UI. | RV is $\mathcal{G}$-meas: $\mathcal{G} \subseteq \mathcal{F}$: if $X^{-1}(A) \in \mathcal{G}$. | not a.s. const: $\exists a$ $\mathbb{P}(X < a) > 0$ & $\mathbb{P}(X \geq a) > 0$
$\mathbb{E}[X] = \int_\Omega X\, d\mathbb{P}$ | RV: $X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ st $\forall A \in \mathcal{E}$, $X^{-1}(A) \in \mathcal{F}$ | remember the $1_{(\ )}$
if μ_p is a Leb-St meas, $\int_\Omega f(t)\, \mu_p(dt) = \int_\Omega f(t) \frac{dF}{dm}(t)\, dt$ where m is the Lebesgue meas
$(\liminf X_n)(\omega) = \liminf(X_n(\omega))$. | $\limsup_n A_n = \limsup_n A_{n+m+1} \in \mathcal{T}_m$ | $(\limsup S_n/n) \in (-\infty, x) = \limsup \frac{X_{m+\ldots} \ldots X_n}{n} \in (-\infty, x)$ as $< m$ finitely $< \infty$.

Candidate Number:

$E[X|\mathcal{G}] := Y$ integ, $\mathcal{G}$-meas st $\forall A\in\mathcal{G}$ $E[Y 1_A] = E[X 1_A] \iff \int_A E[X|\mathcal{G}]\,dP = \int_A X\,dP$ ← defining relation
↺ exists [np] is unique a.s. [$A=\{Y>Y'\}\in\mathcal{G}$ as Y,Y' $\mathcal{G}$-meas, $E[(Y-Y')1_A]=0$ & $(Y-Y')1_A\ge0$ a.s. so $=0$ a.s. $\Rightarrow P(A)=0$ ∘ $Y-Y'>0$ on A] can check ↑ just on π-sys gen $\mathcal{G}$
for partition: $E[X|\sigma(B_n : n\ge1)] = \sum_{n\ge1} E[X 1_{B_n}]/P(B_n)\, 1_{B_n}$ properties, all a.s. 1) $E[E[X|\mathcal{G}]] = E[X]$ $\mathcal{G}=\Omega$ 2) $E[X|\mathcal{G}]=X$ if X $\mathcal{G}$-ms ↺ + on $\{\emptyset\}$
cMCT $X_n\ge0, X_n\uparrow X \Rightarrow E[X_n|\mathcal{G}]\uparrow E[X|\mathcal{G}]$ 3) $E[X|\{\emptyset,\Omega\}]=E[X]$ 4) linear 5) $X\le Y \Rightarrow E[X|\mathcal{G}]\le E[Y|\mathcal{G}] \Rightarrow |E[X|\mathcal{G}]|\le E[|X||\mathcal{G}]$
cFatou $X_n\ge0$, $E[\liminf|\mathcal{G}]\le \liminf E[X_n|\mathcal{G}]$ 6) $E[X|\mathcal{G}]=E[X]$ if $\mathcal{G},\sigma(X)$ indep 7) $E[X|Z]=E[X|\sigma(Z)]$ is $\sigma(Z)$-meas $\Rightarrow$ a fun of Z.
cDCT Y integ, $|X_n|<Y$, $X_n\to X$ a.s. $\Rightarrow E[X_n|\mathcal{G}]\to E[X|\mathcal{G}]$ 5) by lin, just for $X\ge0$: $P(E[X|\mathcal{G}]<0)>0 \Rightarrow P(\{E[X|\mathcal{G}]\le -1/n\})>0$, ev in $\mathcal{G} \Rightarrow$ ※
tower property $X\in L^1$, $\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\mathcal{F}$: $E[E[X|\mathcal{F}_2]|\mathcal{F}_1]=E[X|\mathcal{F}_1]$ a.s. 2 x def rel 6) because E is a meas func?
take out known: X,Y RVs st X,Y,XY integ, Y $\mathcal{G}$-ms: $E[XY|\mathcal{G}]=Y\cdot E[X|\mathcal{G}]$ a.s. start w/ $Y=1_A$, progress through simple, non-neg w/ lin, cMCT
cJensen $X\in L^1$, $E[f(X)]<\infty \Rightarrow E[f(X)|\mathcal{G}]\ge f(E[X|\mathcal{G}])$ a.s. if f convex ⇒ $f=\sup$ family affine funcs g_n, $g_n(E[\cdot])=E[g_n(\cdot)]\le\cdots$
X an RV, $\mathcal{F}_\alpha:\alpha\in I$ are σ-algs, $\mathcal{F}_\alpha\subseteq\mathcal{F}$. $\{X_\alpha := E[X|\mathcal{F}_\alpha] : \alpha\in I\}$ is UI $E[|X_\alpha|]=E[|E[X|\mathcal{F}_\alpha]|]\le E[E[|X|\,|\mathcal{F}_\alpha]]=E[|X|]$
total expectation: $E[X]=\sum_i E[X 1_{A_i}]=\sum_i\int X\, P(d\omega|A_i)P(A_i) \cong \sum E[X|A_i]P(A_i)$ $E[|X_\alpha| 1_{>K}]\le E[|X| 1_{|X_\alpha|>K}]$ by tower, I conv, $\{X\}$ UI
$Cov(X,Y)=E[(X-E(X))(Y-E(Y))]=E[XY]-E[X]E[Y]$. | $\langle X,Y\rangle := E[XY]$ is a scalar prod. orthog if $\langle X,Y\rangle=0 \Rightarrow \|X+Y\|_2^2=\|X\|_2^2+\|Y\|_2^2$
H complete vect space $\subseteq L^2$ $\inf_{Z\in H}\|X-Z\|_2$ attained by $Y\in H$, $(X-Y)$ orthog to H. prf by Cauchy, vect sp Prove cond exp exists: $H=L^2(\Omega,\mathcal{G},P)$. ortho proj
filtration: seq $(\mathcal{F}_n)_{n\ge0}$ of σ-algs, $\mathcal{F}_n\subseteq\mathcal{F}$, $\mathcal{F}_{n+1}$ ∀n $\mathcal{F}_\infty := \sigma(\bigcup_i \mathcal{F}_n)$ is cond exp: start w 1_{A}, simple, cMCT
(X_n) adapted to $\mathcal{F}_n$ if ∀n X_n is $\mathcal{F}_n$-meas integrable if ∀n X_n integ. $\mathcal{F}_n^X := \sigma(X_0,X_1,\dots X_n)$. hitting time of Borel set B
stopping time: τ, RV $\Omega\to\mathbb{N}\cup\{\infty\}$ st ∀n, $\tau\le n/\ge n/=n/<n/>n \in\mathcal{F}_n$ – e.g. const, min, max of stopping times ↗
$\mathcal{F}_\tau := \{A\in\mathcal{F}_\infty : \forall n\ge0\ A\cap\{\tau=/\ge/\le/</>n\}\in\mathcal{F}_n\}$ ← is a σ-alg, $\tau\le\rho \Rightarrow \mathcal{F}_\tau\subseteq\mathcal{F}_\rho$. [$A\in\mathcal{F}_\tau \Rightarrow A\cap\{\rho\le t\}=A\cap\{\rho\le t\}\cap\{\tau\le t\}$]
stopped process of X_n, τ: $X^\tau := (X_{\tau\wedge n})_{n\ge0}$ is adapted to $\mathcal{F}_{\tau\wedge n}\subseteq\mathcal{F}_n$ prf: for X_ρ is $\mathcal{F}_\rho$ meas if $\rho<\infty$: $\{X_\rho\le x\}\cap\{\rho=n\}=\{X_n\le x\}\cap\{\rho=n\}$
X_ρ: $X_\rho(\omega)=X_{\rho(\omega)}(\omega)$, $X_\rho^{-1}(B)=\bigcup_{n\ge0}\rho^{-1}(\{n\})\cap X_n^{-1}(B)\in\mathcal{F}_\rho$ ✓. $\in\mathcal{F}_n$ ∀n.

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martingale: $(X_n)_{n\ge0}$, integ, $\mathcal{F}_n$ adapted, ∀n $E[X_{n+1}|\mathcal{F}_n]=X_n$, $\ge/\le$ if submart/supermart. ext. for $n\ge m$,
$E[X_n]=/\ge/\le E[X_0]$ martingale difference seq: $(Y_n)_{n\ge1}$ integ, adapted $E[Y_{n+1}|\mathcal{F}_n]=0$ a.s. | $f(X_n)$ is a submart if f is convex on $\mathbb{R}$, X a mart cJensen
predictable: $(V_n)_{n\ge1}$ st V_n is $\mathcal{F}_{n-1}$ meas martingale transform: V_n predictable, X_n a mart, $(V\circ X)_n := \sum_{k=1}^n V_k(X_k-X_{k-1})$ is a mart | $V_n\ge0$, X_n supermart ⇒ $V\circ X$ super
Doob's Decomposition X_n integ, adapted: $X_n=X_0+M_n+A_n$, M_n a mart, A_n predict. uniq: $P(M_n=\tilde M_n, A_n=\tilde A_n\ \forall n)=1$ | X_n submart ⟺ $A_{n+1}\ge A_n$ ∀n
↳ $A_n := \sum_{k=1}^n E[X_k-X_{k-1}|\mathcal{F}_{k-1}]=\sum^n E[X_k|\mathcal{F}_{k-1}]-X_{k-1}$, $M_n=\sum_{k=1}^n(X_k-E[X_k|\mathcal{F}_{k-1}])$ uniq + sub/super: $E[X_{n+1}-X_n|\mathcal{F}_n]=A_{n+1}-A_n$. supermart ⟺ $A_{n+1}\le A_n$ ∀n
$\langle M\rangle_n$: M a L^2 mart ⟺ $E[M_n^2]<\infty$, ⇒ M^2 a submart, Doob decomp, $\langle M_n\rangle = A_n$, which is incr.
stopped martingale: $X_{\tau\wedge n}$ is a mart wrt $\mathcal{F}_{\tau\wedge n}$, $\mathcal{F}_n$ $V_k=1_{k\le\tau}$ is predict as $\{\tau\ge k\}=\{\tau>k-1\}$. $X_{\tau\wedge n}=X_0+(X\circ V_k)_n$
Doob OST τ,ρ bdd st's, $\tau\le\rho$ a) $E[X_\rho|\mathcal{F}_\tau]=X_\tau$ a.s., b) $E[X_\rho]=E[X_\tau]=E[X_0]$. a) $\rho=n$: check def rel, split on $\tau=k$, mart prop
a) general: $V_k=1_{\rho\ge k>\tau}$ is pred & bdd, $V\circ X$ = mart, $V_k=1_{\rho\ge k}-1_{\tau\ge k} \Rightarrow (V\circ X)_n=X_{\rho\wedge n}-X_{\tau\wedge n}$, $0=(V\circ X)_{\tau\wedge n}=E[(V\circ X)_n|\mathcal{F}_{\tau\wedge n}]$ ∘ $\tau\le\rho$
variants: τ a.s. finite st [i.e. $P(\tau<\infty)=1$], if 1 of: a) $\{M_n : n\ge0\}$ UI b) $E[\tau]<\infty$, ∃L st $E[|M_{n+1}-M_n||\mathcal{F}_n]\le L$ as ∀n. take $n\ge\rho\ge\tau$: take $\rho=n$, $\tau=\tau\wedge n$ $=E[X_{\rho\wedge n}|\mathcal{F}_{\tau\wedge n}]-X_{\tau\wedge n}$
then $E[M_\tau 1_{\tau<\infty}]=E[M_0]$. prf a) $E[M_{\tau\wedge n}]=E[M_0]$, $M_{\tau\wedge n}\to M_\tau 1_{\tau<\infty}$ a.s., so in L^1 by UI, Vitali. $E[\sum^\infty 1_{\tau\ge i}|M_i-M_{i-1}|]=\sum E[\sim]$ by MCT
prf b) WLOG $M_0=0$, $|M_{n\wedge\tau}|=|M_{n\wedge\tau}-M_{0\wedge\tau}|\le\sum_{i=1}^n|M_{i\wedge\tau}-M_{(i-1)\wedge\tau}|\le\sum^\infty|\cdot-\cdot|=\sum^\infty 1_{\tau\ge i}|M_i-M_{i-1}|$. $=\sum$ tower prop $=\sum^\infty E[1_{\tau\ge i}E[|M_i-M_{i-1}||\mathcal{F}_{i-1}]]$
Doob's maximal inequality X_n submart, $\lambda>0$, $Y_n^\lambda := (X_n-\lambda)1\{\max_{k\le n}X_k\ge\lambda\}$ is a submart $\le L\sum E[1_{\tau\ge i}]=L\,E[\tau]<\infty$ $\{\tau\ge i\}\in\mathcal{F}_{i-1}$
& $\lambda P[\max_{k\le n}X_n\ge\lambda]\le E[X_n 1\{\max_{k\le n}X_k\ge\lambda\}]\le E[|X_n|]$. $\tau=h_{[\lambda,\infty)}$, $V_n=1_{\tau\le n-1}$ $\bar X_n=\max_{k\le n}X_k$, $V_n=1_{\bar X_{n-1}\ge\lambda}$
$-X_n$ a supermart, so $V\circ X$ a submart, $(V\circ X)_n=X_{n\vee\tau}-X_\tau=(X_n-X_\tau)1_{\tau\le n}$, $X_\tau\ge\lambda$ by def, $(X_\tau-\lambda)1_{\tau\le n}$ adapted, integ, non-decr ⇒ submart
⇒ Y^λ = sum of 2 submarts = submart, $0\le E[(X_0-\lambda)1_{X_0\ge\lambda}]=E[Y_0^\lambda]\le E[Y_n^\lambda]=E[(X_n-\lambda)1_{\tau\le n}]=E[X_n 1_{\bar X_n\ge\lambda}]-\lambda P(\bar X_n\ge\lambda)$.
$p\ge1$, $M_n\in L^p$, a mart ⇒ $P(\max_{n\le N}|M_n|\ge\lambda]\le E[|M_N|^p]/\lambda^p$ $|M_n|^p$ a submart, raise both sides to p. + Hölder appl L44 P1
Doob's L^p inequality: $p>1$, $X_n\ge0$, submart, $X_n\in L^p$ $\max_{k\le n}X_n\in L^p$, $E[X_n^p]\le E[\max_{k\le n}X_k^p]\le(p/(p-1))^p E[X_n^p]$ Doob max
maximal ineq variant: X_n supermart: $\lambda P(\max_{k\le n}|X_k|\ge\lambda)\le E[X_0]+2E[X_n^-]$ ∀λ,n Doob OST to X, τ: $\min\{k : X_k\ge\lambda\}\wedge n$: $E[X_0]\ge E[X_\tau]$ / Also:
Upcrossings: $(x_n)\in\mathbb{R}$, $a<b$ fixed: $(\rho_k)_{k\ge1}$, $(\tau_k)_{k\ge0}$ st $\tau_0=0$, $\tau_k=\inf\{n\ge\rho_k : x_n\ge b\}$ | $\ge\lambda P(\max_{k\le n}X_k\ge\lambda)+E[X_n 1_{\max<\lambda}]$ ≥0, submart
upcrossings before n $U_n([a,b],x)=\max\{k : \tau_k\le n\}$ | $\rho_k=\inf\{n\ge\tau_{k-1} : x_n\le a\}$ | then $-E[X_n 1_{\max<\lambda}]\le E[X_n^-]$. $\lambda P(\max X_k^-\ge\lambda)\le E[X_n^-]$
total ↑ $U([a,b],x)=\sup_n U_n=\sup\{k : \tau_k<\infty\}$. $E[U_n([a,b],X)]\le\frac{E[(X_n-a)^-]}{b-a}$ ρ_k,τ_k stopping times, $V_n=\sum_{k\ge1}1_{\rho_k<n\le\tau_k}$ = 0 or 1.
Doob upcrossing: X supermart, fix $a<b$, ∀n≥0 so $V_n\ge0$, predict, $(V\circ X)_n\ge(b-a)U_n-(X_n-a)^-$, $(V\circ X)_0:=0$.
(x_n) converges in $[-\infty,\infty]$ ⟺ $U([a,b],x)<\infty$ ∀ $a<b\in\mathbb{Q}$ | X_n bdd in L^p if $\sup_n E[|X_n|^p]<\infty$.
Doob Forward Conv: X sub/supermart, X_n bdd in L^1 then $X_n\to X_\infty$ a.s., X_∞ integ. WLOG supermart, e.g. use $(-X_n)$. fix $a<b$, Doob UP
$E[U_n([a,b],X)]\le E[(X_n-a)^-]/(b-a)\le(E[|X_n|]+|a|)/(b-a)$. $U_n\uparrow U$, by MCT: $E[U(\cdot)]\le(\sup_n E|X_n|+a)/(b-a)<\infty \Rightarrow P(U=\infty)=0$ so $X_n\to X_\infty$ a.s.
Let $X_\infty := \liminf X_n$, is integ by Fatou. $X_n\not\to X_\infty$ in L^1 !! ⇒ $P(\exists a,b\in\mathbb{Q}$ st $U([a,b])=\infty)=0$ ∘ $\mathbb{Q}$ countable by red conts
M_n a mart: TFAE: 1) M is UI 2) ∃ M_∞ st $M_n\to M_\infty$ a.s., in L^1 3) ∃ M_∞ st $M_n=E[M_\infty|\mathcal{F}_n]$ a.s. ∀n | so X_∞ exists a.s. for non-neg. supermarts
1)⇒2): UI ⇒ bdd, Doob Conv, Vitali for L^1 conv ↺ $\mathcal{F}_\infty$-measurable 2⇒3) check M_n for def rel: M_n is $\mathcal{F}_n$-meas, $A\in\mathcal{F}_n$: $E[M_\infty 1_A]=E[M_n 1_A]$ $m\ge n$ no proof
↺ setup as on L48, ∀ stopping time [even $=\infty$] $\tau\le\rho$ $E[M_\rho|\mathcal{F}_\tau]=M_\tau$ a.s. 3⇒1) by L10 P2. $|E[(M_\infty-M_m)1_A]|\le E[|M_\infty-M_m|1_A]\le E[|M_\infty-M_m|]\to0$
& Doob ineqs for ∞: $M_\infty^* := \max_{n\ge0}|M_n|$ $\lambda P(M_\infty^*\ge\lambda)\le E[|M_\infty|1\{M_\infty^*\ge\lambda\}]$ $\lambda\ge0$, $M_\infty\in L^p$ for $p>1 \Rightarrow \|M_\infty\|_p\le\|M_\infty^*\|_p\le q\|M_\infty\|_p$, $M_n\to M_\infty$ in L^p
Backwards: time indexed by $I=\mathbb{Z}_{\le0}$, ends at 0. filtrations: $(\mathcal{F}_{-n})_{n\ge0}$: ∀n $\mathcal{F}_n\subseteq\mathcal{F}$, $\mathcal{F}_{-n}\subseteq\mathcal{F}_{-n+1}$.
backwards martingale: ∀n M_{-n} integ, $\mathcal{F}_{-n}$ meas, $E[M_{-n+1}|\mathcal{F}_{-n}]=M_{-n}$ a.s. | $M_{-n}=E[M_0|\mathcal{F}_{-n}] \Rightarrow M_n$ UI [Lie 48]
Doob's Upcrossing holds (as for finite num of marts) for U_n = # crossings in $[-n,0]$ | $M_{-n}\to M_{-\infty}=\liminf M_n$ by Doob Forward Conv
$\mathcal{F}_{-\infty} := \bigcap_{k=0}^\infty\mathcal{F}_{-k}$ as $k\uparrow$ $\mathcal{F}_{-k}\downarrow$. M_∞ is $\mathcal{F}_{-\infty}$ meas. $=E[M_0|\mathcal{F}_{-\infty}]$ and conv is a.s., in L^1
Kolmogorov's SLLN: (X_n) iid, integrable, mean m, $S_n=\sum_{i=1}^n X_i$: $S_n/n\to m$ a.s. & in L^1. $\mathcal{F}_{-n} := \sigma(S_n,S_{n+1},\dots)=\sigma(S_n,X_{n+1},X_{n+2},\dots)$
$E[X_1|\mathcal{F}_{-n}]=E[X_i|\mathcal{F}_{-n}]$ for $i\le n$ as $X_1\dots X_n$ symmetric under $\mathcal{F}_{-n}$. so $E[X_1|\mathcal{F}_{-n}]=\frac{1}{n}S_n$. $M_{-n}=S_n/n$: $E[M_{-n+1}|\mathcal{F}_{-n}]=E[S_{n-1}/(n-1)|\mathcal{F}_{-n}]$
so M_{-n} backwards mart so S_n/n conv a.s. & in L^1 to $M_{-\infty}$. L^1 conv says $E[M_{-\infty}]=E[M_{-1}]=m$ & const a.s. by KO-1. $=\frac{1}{n-1}\sum^{n-1}E[X_i|\mathcal{F}_{-n}]=\frac{1}{n-1}\cdot\frac{n-1}{n}S_n=M_{-n}$