Integration

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Note:

L'Hopital's rule(s): Given $f, g : \mathbb{R} \to \mathbb{R}$ are defined and continuous on some closed interval I containing a point a (a may be an endpoint of the interval), and differentiable on $\text{int}(I)\backslash\{a\}$, with $f'(a), g'(a)$ existing, and $f(a) = g(a) = 0$, $g'(a) \neq 0$, then

$$\lim_{x \to a/a_-/a_+} \frac{f(x)}{g(x)} = \lim_{x \to a/a_-/a_+} \frac{f'(a)}{g'(a)}$$

(Choose limit as appropriate to the interval)

Weirder bounds: use Taylor's theorem to bound difficult functions - write out the Taylor expansion, and then bound that instead.

Don't forget measurability !!!!!!!

proving not integrable: can choose just a subset of the region - e.g. to prove $\frac{xy}{(x^2+y^2)^3}$ is not integrable over $(-1, 1) \times (-1, 1)$, show it isn't over $(0, 1) \times (0, 1)$ (which then allows for easy application of Tonelli's theorem)

Aside: the problems with the Riemann integral

it works for continuous functions, other Riemann integrable ones, but it doesn't work on everything:

e.g. $f = \mathbf{1}_{\mathbb{Q}\cap[0,1]} = \chi_{\mathbb{Q}\cap[0,1]}$, for which $\sup_{\phi_-} I(\phi_-) = 0$ and $\sup_{\phi_+} I(\phi_+) = 1$ so this function isn't integrable, even though it would be nice to define the length of a set as $m(E) = \int \chi_E(x)dx$.

Also, we lack convergence for indefinite integrals - there's nothing that says $f_n \to f \implies \int f_n(x)dx \to \int f(x)dx$, but we do have that if $f_n \to f$ uniformly on $[a, b]$ then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$

We also can't use it for probability theory and summing infinite series.

1 Basics

using $\mathbb{R}_\infty = [-\infty, \infty]$ with multiplication, addition as expected, so any $E \subseteq [-\infty, \infty]$ has a supremum and infimum in $[-\infty, \infty]$. (note $\sup(\emptyset) = -\infty$)

for absolutely convergent/non-negative series you can sum in any order:

$\sum_{n=1}^{\infty} a_n = \sup\left\{\sum_{n \in J} a_n : J \text{ a finite subset of } \mathbb{N}\right\}$

And for a 2D series you can sum it whichever way you like, and extend the sup of finite subsets idea.

$\limsup_{n\to\infty} a_n = \lim_{m\to\infty}\left(\sup_{n\geq m} a_n\right)$, $\liminf_{n\to\infty} a_n = \lim_{m\to\infty}\left(\inf_{n\geq m} a_n\right)$

Useful properties: [1.3, obvious/sheet 1 q5]

- $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)$,
- $\liminf \leq \limsup$
- limit exists $\iff$ $\limsup = \liminf$,
- $\limsup$ preserves weak inequalities, follows triangle law.

2 Lebesgue measure

required properties for $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ to be a **measure** on $\mathbb{R}$:

(i) $m(\emptyset) = 0, m(\{x\}) = 0$

(ii) $m(I) = b - a$ where I is an interval with endpoints $a < b$

(iii) $m(A + x) = m(A)$

(iv) $m(\alpha A) = |\alpha| m(A)$

(v) $m(A) \leq m(B)$ if $A \subseteq B$ (is *monotone*)

(vi) $m(A \cup B) = m(A) + m(B)$ if $A \cap B = \emptyset$ (is *finitely additive*)

   (a) $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$ if they are all disjoint from one another (*countably additive*)

(vii) $m(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$ if (A_n) is an increasing sequence of sets

Note 6. $\implies$ $m(A \backslash B) = m(A) - m(B)$.

the **Lebesgue outer measure** is $m^*(A) = \inf\left\{\sum_{n=1}^{\infty} m(I_n) : I_n \text{ intervals}, A \subseteq \bigcup_{n=1}^{\infty} I_n\right\}$ for $A \subseteq \mathbb{R}$.

Properties thereof:

(i) $m^*(\emptyset) = 0, m^*(\{x\}) = 0$

(ii) $m^*(I) = b - a$ where I is an interval with endpoints $a < b$

(iii) $m^*(A + x) = m^*(A)$

(iv) $m^*(\alpha A) = |\alpha| m^*(A)$

(v) $m^*(A) \leq m^*(B)$ if $A \subseteq B$ (is *monotone*)

(vi) $m^*(A \cup B) \leq m^*(A) + m^*(B)$ if $A \cap B = \emptyset$ (is partly *finitely additive*)

  (a) $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$ if they are all disjoint from one another (*countably subadditive,* proof in 2015)

$E \subseteq \mathbb{R}$ is **null** if $m^*(E) = 0$. Also null:

- any subset of a null set
- countable union of null sets
- countable subsets of $\mathbb{R}$ are all null - e.g. $\mathbb{N}, \mathbb{Q}$.
- the **Cantor set** is null and closed. Its two definitions:
  - let $C_0 = [0, 1]$, $C_{n+1} = C_n/3 \cup (C_n + 2)/3$ - i.e. taking out the middle third each time, and $C = \bigcap\limits_{n=1}^{\infty} C_n$
  - $C = \left\{ x \in [0, 1] : \exists (a_n)_{n \geq 1} \in \{0, 2\}^{\mathbb{N}} \text{ st } x = \sum\limits_{n=1}^{\infty} a_n 3^{-n} \right\}$
  - both are equivalent (kinda annoying to see)
  - it has measure 0, by using the set definition
  - the **Cantor-Lebesgue function** $\Phi : [0, 1] \to [0, 1]$ is $\Phi(x) = \sum\limits_{n=1}^{\infty} a_n/2 \cdot 2^{-n}$ where the a_n are defined as above for $x \in C$, and $\Phi(y) = \sup_{x \geq y, x \in C} \Phi(x)$.
    - this is discussed in detail in 2019Q1c, and is measurable.

a property Q of real numbers holds **almost everywhere** if the set of reals it does not hold on is null.

m^* is not countably additive on $\mathbb{R}$ [if it was a Vitali set would be measurable - $A \subseteq [0, 1]$ st $x, y \in A, x \neq y \implies x - y \notin \mathbb{Q}$, $\forall x \in [0, 1]$ $\exists q \in \mathbb{Q}$ st $x + q \in A$ taking $\bigcup_{q \in \mathbb{Q} \cap [0,1]} (A - q)$,...]

$E \subseteq \mathbb{R}$ is **Lebesgue measurable** if $m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$ for all $A \subseteq \mathbb{R}$ - note $A \backslash E := A \cap (\mathbb{R} \backslash E)$, and we automatically have $m^*(A) \leq m^*(A \cap E) + m^*(A \backslash E)$

let $\mathcal{M}_{\text{Leb}}$ be the set of Lebesgue measurable sets. It contains: [proofs, see Capinski & Kopp]

- null sets
- intervals
- $\mathbb{R} \backslash E \in \mathcal{M}_{\text{Leb}}$ if $E \in \mathcal{M}_{\text{Leb}}$

- $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}_{\text{Leb}}$ if $E_n \in \mathcal{M}_{\text{Leb}}$, and if $E_n \cap E_k = \emptyset$ for all $n \neq k$ then $m^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m^*(E_n)$
- open and closed subsets of $\mathbb{R}$ [as open sets a countable union of intervals]

for $E \in \mathcal{M}_{\text{Leb}}$ write $m(E) = m^*(E)$, so m is countably additive (on $\mathcal{M}_{\text{Leb}}$), and m satisfies all the properties of a measure for $A, B, A_n \in \mathcal{M}_{\text{Leb}}$:

(i) $m^*(\emptyset) = 0, m^*(\{x\}) = 0$

(ii) $m^*(I) = b - a$ where I is an interval with endpoints $a < b$

(iii) $m^*(A + x) = m^*(A)$

(iv) $m^*(\alpha A) = |\alpha| m^*(A)$

(v) $m^*(A) = m^*(B)$ if $A \subseteq B$ (is *monotone*)

(vi) $m^*(A \cup B) = m^*(A) + m^*(B)$ if $A \cap B = \emptyset$ (is partly *finitely additive*) $\implies m^*(A \backslash B) = m^*(A) - m^*(B)$

   (a) $m^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m^*(A_n)$ if they are all disjoint from one another (*partly countably additive*)

3 Measure spaces, measurable functions

given a set Ω, $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is **σ-algebra/σ-field** on Ω if :

(i) $\emptyset \in \mathcal{F}$

(ii) if $E \in \mathcal{F}$ then $\Omega \backslash E \in \mathcal{F}$

(iii) if $E_n \in \mathcal{F}$ for $n = 1, 2, ...$ then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$
$\implies \bigcap_{n=1}^{\infty} E_n \in \mathcal{F}$

if so, then $(\Omega, \mathcal{F})$ is a **measurable space**, and sets in $\mathcal{F}$are **$\mathcal{F}$-measurable**.

a **measure** on $(\Omega, \mathcal{F})$ is a function $\mu : \mathcal{F} \to [0, \infty]$ st

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A) \leq \mu(B)$ if $A \subseteq B, A, B \in \mathcal{F}$

(iii) $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ if the $E_n$are disjoint sets in $\mathcal{F}$

then $(\Omega, \mathcal{F}, \mu)$ is **measure space**. a measure μ is **finite** if $\mu(\Omega) < \infty$, and a **probability measure** if $\mu(\Omega) = 1$

examples:

- $(\mathbb{R}, \mathcal{M}_{\text{Leb}}, m)$,
- the counting measure: $(\Omega, \mathcal{P}(\Omega), \mu = E \mapsto |E|)$ for any set Ω
- $([0, 1], \mathcal{M}_{\text{Leb}}|_{[0,1]}, m)$ - a probability measure
- $(\Omega, \mathcal{F}, \mathbb{P})$, as defined in probability, and thus a probability measure
- the **Lebesgue-Stieltjes measure for** $F : \mathbb{R} \to \mathbb{R}$, an increasing function, assumed that $\forall x\ F(x) = \lim_{y \to x+} F(y)$:
  - $m_F^*(E) = \inf \left\{ \sum\limits_{n=1}^{\infty} m_F(J_n) : J_n = (a_n, b_n], E \subseteq \bigcup\limits_{n=1}^{\infty} J_n \right\}$
  - acting on a $\sigma-$algebra $\mathcal{M}_F$ containing all intervals, where $m_F(a, b] = F(b) - F(a)$
  - so, m_F^* acts similarly to m^* except that $m_F^*(a, b) = F(b-) - F(a); m_F^*([a, b]) = F(b) - F(a-); m_F^*(\{x\}) = 0 \iff F$ is cont at x

useful properties of a measure space $(\Omega, \mathcal{F}, \mu)$:

(i) for $A, B \in \mathcal{F}$ with $A \subseteq B$, $\mu(A) \leq \mu(B)$ [prove with disjoint union]

(ii) for a sequence $(A_n) \in \mathcal{F}$ with $A_n \subseteq A_{n+1}$ then $\mu(\bigcup_n A_n) = \lim_{n \to \infty} \mu(A_n)$ [prove with $A'_r = A_r \backslash A_{r-1}$]

(iii) for a sequence $(A_n) \in \mathcal{F}$ with $A_n \supseteq A_{n+1}$ and $\mu(A_1) < \infty$ then $\mu(\bigcap_n A_n) = \lim_{n \to \infty} \mu(A_n)$ [prove with (ii), take complements, consider $\mu(\Omega)$]

given $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ there is a unique σ-algebra $\mathcal{F}_\mathcal{B}$ on Ω generated by $\mathcal{B}$ in the sense that $\mathcal{B} \subseteq \mathcal{F}_\mathcal{B}$, and if $\mathcal{F}$ is another σ-algebra on Ω with $\mathcal{B} \subseteq \mathcal{F}$ then $\mathcal{F}_\mathcal{B} \subseteq \mathcal{F}$ [like a closure/interior

$\mathcal{M}_{\text{Bor}}$ is the **Borel $\boldsymbol{\sigma}$-algebra on** $\mathbb{R}$ is the algebra on $\mathbb{R}$ generated by the intervals:

- description: "the class of subsets of $\mathbb{R}$ constructable from intervals in a countable number of complements, countable unions, or countable intersections".
- $\mathcal{M}_{\text{Bor}}$ is the smallest σ-algebra on $\mathbb{R}$ containing (†)
  - (i) all intervals
  - (ii) (a, ∞) for all $a \in \mathbb{R}$
  - (iii) all closed intervals
  - (iv) all open sets
- $\mathcal{M}_{\text{Bor}} \neq \mathcal{M}_{\text{Leb}}$ [no need to prove]
- if $E \in \mathcal{M}_{\text{Leb}}$ there exist $A, B \in \mathcal{M}_{\text{Bor}}$ st $A \subseteq E \subseteq B$ and $B \backslash A$ is null (so $E \backslash A$ and $B \backslash E$ are also null) [no need to prove, in textbook]

the **push-forward σ-algebra of $\boldsymbol{\mathcal{F}}$ by $\boldsymbol{f}$** is $f_*(\mathcal{F}) := \{G \subseteq \mathbb{R} : f^{-1}(G) \in \mathcal{F}\}$ for a function $f : \Omega \to \mathbb{R}$ and a σ-algebra $\mathcal{F}$. It is a σ-algebra over $\mathbb{R}$ [$f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\mathbb{R} \backslash G) = \Omega \backslash f^{-1}(G)$,...]

a function $f : \Omega \to \mathbb{R}$, given $(\Omega, \mathcal{F})$ is measurable, is **$\boldsymbol{\mathcal{F}}$-measurable**

$\iff$ the set of all intervals $\mathcal{I} \subseteq f_*(\mathcal{F}) \iff \forall$ intervals $I \in \mathbb{R}$, $f^{-1}(I) \in \mathcal{F}$

$\iff$ $\mathcal{M}_{\text{Bor}} \subseteq f_*(\mathcal{F}) \iff \mathcal{B} \subseteq f_*(\mathcal{F})$ where $\mathcal{B}$ is one of the sets in (†)
[proof since $\mathcal{I}$ is one of the $\mathcal{B}$, and for all of those $\mathcal{B}$, $\mathcal{B} \subseteq \mathcal{M}_{\text{Bor}}$, and if $\mathcal{B} \subseteq f_*(\mathcal{F})$ so is $\mathcal{F}_\mathcal{B} = \mathcal{M}_{\text{Bor}}$].

Various Lebesgue-measurable functions:

- constant functions,
- characteristic functions $\chi_A$of a set $A \iff A$ is measurable
- continuous or monotone functions $f : \mathbb{R} \to \mathbb{R}$
- functions continuous a.e.
- $g = f$ a.e. if $f : \mathbb{R} \to \mathbb{R}$ is measurable
- RVs in probability
- any function that can be explicitly defined
- $f + g, fg, \max(f, g)$ for $f, g : \mathbb{R} \to \mathbb{R}$, both measurable
  [e.g. $(f + g)^{-1}((a, \infty)) = \bigcup_{q \in \mathbb{Q}} f^{-1}(q, \infty) \cap g^{-1}(a - q, \infty)$ which is measurable]
- $h \circ f$ for f measurable, h continuous is Borel measurable
  [as if $G \subseteq \mathbb{R}$ is open then $h^{-1}(G)$ is, so $f^{-1}(h^{-1}(G))$ is measurable]

a function $f : \mathbb{R} \to [-\infty, \infty]$ is **measurable**

$\iff$ $\forall a \in \mathbb{R}\ f^{-1}(a, \infty] \in \mathcal{M}_{\text{Leb}}$

$\iff$ $\big(\forall B \in \mathcal{M}_{\text{Bor}}\ f^{-1}(B) \in \mathcal{M}_{\text{Leb}}\big) \wedge f^{-1}(\{\infty\}) \in \mathcal{M}_{\text{Leb}}$

$\iff$ $\arctan \circ f$ is measurable where $\arctan : [-\infty, \infty] \to [-\pi/2, \pi/2]$ is the inverse tan function.

given a sequence (f_n) of measurable functions $\mathbb{R} \to [-\infty, \infty]$, then the following are measurable:

- $\sup_n f_n,\ \inf_n f_n$ [prove $(\sup_n f_n)^{-1}(a, \infty] = \bigcup_{n\in\mathbb{N}} f_n^{-1}(a, \infty]$ by double incl]
- $\limsup_{n\to\infty} f_n,\ \liminf_{n\to\infty} f_n$ [prove using $\limsup f_n = \inf_m \big(\sup_{n\geq m} f_n\big)$]
- $\lim_{n\to\infty} f_n$, if it exists [by $\limsup, \liminf$]

a function $\phi : \mathbb{R} \to \mathbb{R}$ is **simple** if it is measurable and takes finitely many real values. e.g.:

- χ_E if $E \in \mathcal{M}_{\text{Leb}}$,
- $\phi + \psi,\ \phi\psi,\ \alpha \cdot \phi,\ \max(\phi, \psi),\ h \circ \phi$ for ϕ, ψ simple, h any function
- any function of the form $\sum_{j=1}^{n} \beta_j X_{E_j}$ for $\beta_j \in \mathbb{R}$, $E_j \in \mathcal{M}_{\text{Leb}}$
- step functions [but simple functions are not always step functions]

if $\phi = \sum_{i=1}^{k} \alpha_i \chi_{B_i}$ where ϕ takes non-zero values $\alpha_1, \alpha_2, ..., \alpha_k$ and $B_i = \phi^{-1}(\{\alpha_i\})$ then ϕ is in **standard/canonical** form. - e.g. the standard from of $\chi_{(0,2)} + \chi_{[1,3]}$ is $1 \cdot \chi_{(0,1)\cup[2,3]} + 2 \cdot \chi_{[1,2)}$

for a measurable function $f : \mathbb{R} \to [0, \infty]$ there is an increasing sequence (ϕ_n) of non-negative *simple* functions st $f(x) = \lim_{n\to\infty} \phi_n(x)$ for all $x \in \mathbb{R}$.

[3.9, proof: $B_{k,n} = \{x : f(x) \in [k2^{-n}, (k+1)2^{-n})\}$ for $n = 1, 2, ...; k = 0, 1, 2, .., 4^n - 1$ and $\phi_n(x) = k2^{-n}$ if $x \in B_{k,n}$for some (unique) k, otherwise 2^n if $f(x) \geq 2^n$]

$f : \mathbb{R} \to \mathbb{R}$ is measurable $\iff$ there is a sequence (ψ_n) of step functions st $f = \lim \psi_n$ <u>a.e.</u> [textbook]

4 The Lebesgue integral

Definition for non-negative simple functions: for $\phi = \sum_{i=1}^{k} \alpha_i \chi_{B_i}$ (i.e. $\alpha_i > 0$) in standard form,

$$\int_{\mathbb{R}} \phi = \int_{-\infty}^{\infty} \phi(x)dx = \sum_{i=1}^{k} \alpha_i m(B_i)$$

Which is finite $\iff \forall i\ m(B_i) < \infty$. This definition also works for non-negative simple functions not in standard form.

Useful notes [4.1]: $\int(\phi+\psi) = \int \phi + \int \psi$ for ϕ, ψ non-negative simple functions, $\int \alpha\phi = \alpha \int \phi$ for $\alpha \in [0, \infty)$

if $\phi \leq \psi$ pointwise then $\int \phi \leq \int \psi$

Definition for non-negative measurable functions: for $f : \mathbb{R} \to [0, \infty]$

$$\int_{\mathbb{R}} f = \sup \left\{ \int_{\mathbb{R}} \phi : \phi \, \text{simple}, 0 \leq \phi \leq f \right\}$$

And its integral over a measurable set $E \subseteq R$ is $\int_E f = \int_{\mathbb{R}} f \chi_E$ if f is defined over all of $\mathbb{R}$, and otherwise, if $f : E \to [0, \infty]$ then $\int_E f = \int_{\mathbb{R}} \tilde{f}$, where $\tilde{f} = f$ on E and $= 0$ everywhere else.

f is **integrable** over $E \subseteq \mathbb{R}$ if $\int_E f < \infty$.

Clearly if $f \leq g$, $\int f \leq \int g$, and $\int \alpha f = \alpha \int f$ for $\alpha \geq 0$

MCT v1 [4.2] if (f_n) is an increasing sequence of non-negative measurable functions, and $f = \lim_{n\to\infty} f_n = \sup_n f_n$, then $\int f = \lim_{n\to\infty} \int f_n$.

Proof:

- $\forall n\ f_n \leq f$, so $\sup_n \int f_n \leq \int f$
- for $\lim_{n\to\infty} \int f_n \geq \int f$, we want to find ϕ st $0 \leq \phi \leq f$, $\int \phi \leq \lim_{n\to\infty} \int f_n$, so that, $\lim_{n\to\infty} \int f_n \geq \int f$ as $\int f$ is a supremum.
  - defining B_n: for $\alpha \in (0, 1)$,
    - let $B_n = \{x : f_n(x) \geq \alpha\phi(x)\}$, which is measurable, as $f_n - \alpha\phi$ is.
    - $B_n \subseteq B_{n+1}$, $\cup_{n=1}^{\infty} B_n = \mathbb{R}$, cause $\lim_{n\to\infty} B_n = \mathbb{R}$
    - (†) $\alpha \int_{B_n} \phi \leq \int_{\mathbb{R}} f$ as $\alpha\phi\chi_{B_n} \leq f_n\chi_{B_n} \leq f_n$
  - given $\phi = \sum_{i=1}^{k} \beta_i \chi_{E_i}$,
  - $\int_{B_n} \phi = \sum_{i=1}^{k} \beta_i m(E_i \cap B_n) \to \sum_{i=1}^{k} \beta_i m(E_i) = \int_{\mathbb{R}} \phi$ because $\lim_{n\to\infty} B_n = \mathbb{R}$
  - so $\alpha \int_{\mathbb{R}} \phi \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n$ by taking limits in (†)
  - Then let $\alpha \to 1-$

Baby MCT [4.3] given f is a non-negative/non-positive (it must have the same sign over all of E) measurable function, (E_n) an increasing sequence of measurable sets, $E = \cup_{n=1}^{\infty} E_n$, then f is *integrable* over $E \iff \sup_n \int_{E_n} f < \infty$, and then $\int_E f = \sup_n \int_{E_n} = \lim_{n\to\infty} \int_{E_n}$

Proof: use MCT v1 with $f_n = f\chi_{E_n}$.

Adding integrals [4.4] for non-negative measurable functions f, g $\int(f+g) = \int f + \int g$

Proof: take increasing sequences of simple functions for each of f, g, use MCT v1 and the fact that the integrals of simple functions add properly.

MCT for Series [4.5] For a sequence (g_n) of integrable functions that are all non-negative a.e., and the sum of their integrals is finite, then their sum converges a.e. to an integrable function , and $\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n$

Agreeing with the Riemann integral [4.6] For a continuous function $f : [a, b] \to [0, \infty)$, the Lebesgue and Riemann integrals agree. Proof: step functions are simple functions, and there is an increasing sequence of step functions with limit f since it is Riemann integrable, so apply the MCT to them.

This is also true for $f : [a, b] \to [-\infty, \infty]$, since f is Riemann integrable $\iff$ f is bounded and continuous a.e., and in such a case it is also Lebesgue measurable. - see notes page 17 for a brief argument.

Definition for all measurable functions: for $f : \mathbb{R} \to [-\infty, \infty]$, let $f_+ = \max(f, 0)$, $f_- = \max(-f, 0)$, so $f = f_+ - f_-, |f| = f_+ + f_-$.

f is **integrable** $\iff$ both f_+, f_- are, and $\int f = \int f_+ - \int f_-$

Useful facts [4.8]:

1. f is integrable $\implies$ $|f|$ is integrable
2. f is measurable and $|f|$ is integrable $\implies$ f is integrable
3. **Comparison test:**
    - f is measurable, and $|f| \leq g$ for integrable $g \implies f$ is integrable
    - $|f| \geq g \geq 0$ for measurable, non-integrable $g \implies f$ is <u>not</u> integrable
4. The integral is a linear operator on integrable functions (when their sums are defined)
5. $f \leq g \implies \int f \leq \int g$
6. f integrable, $f = g$ a.e. $\implies$ g is integrable, $\int g = \int f$. <u>Note</u>: this means the integral over an interval is the same as that over its closure, etc.
7. f integrable $\implies f(x) \in \mathbb{R}$ a.e.
8. f integrable, $\int |f| = 0 \implies f = 0$ a.e.

9. f integrable over measurable $E = \cup_{n=1}^{\infty} E_n$, for measurable E_n, $\int_E f = \lim_{n\to\infty} \int_{E_n} f$

Proofs: 1), 2) from def of $\int |f|$;

3) since $|f| \leq g \implies \int |f| \leq \int g$;

4),5) by splitting into $+, -$;

6) since $|f-g| = 0$ a.e., so any non-negative step function smaller than $|f-g|$ is 0 a.e., so the integral is 0;

7) by contradiction/ see s2q9;

8) s2q9 or directly since $f_+, f_- \leq |f|$, same argument as 6;

9) by applying Baby MCT to f_+, f_-

Extensions to the Comparison test [4.9]:

- g integrable, h bounded and measurable $\implies$ hg is integrable [prove using $|gh| \leq C|g|$]
- g integrable over $\mathbb{R}$ $\implies$ g integrable over any measurable subset of $\mathbb{R}$
- h bounded and measurable is integrable over any subset with finite measure.

Fundamental Theorem of Calculus (FTC) [4.1]: if g is a function with a continuous derivative on a closed bounded interval $[a, b]$, then g' is integrable over $[a, b]$ and $\int_a^b g'(x)dx = g(b) - g(a)$

No proof necessary, as Riemann=Lebesgue for such g.

Integration by parts [4.13]: For f, g continuously differentiable [i.e. have continuous derivative] on a closed bounded interval $[a, b]$, then

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

Again, proof by Prelims.

Substitution [4.15]: for a monotonic function $g : I \to \mathbb{R}$ with a continuous derivative, let $J = g(I)$, so J is an interval. A measurable function $f : J \to \mathbb{R}$ is integrable $\iff (f \circ g) \cdot g'$ is integrable over I.

$$\int_j f(x)dx = \int_I f(g(y))|g'(y)|dy$$

Note neither I nor J must be closed or bounded.

Proof: left out, see Qian 7.4

4.1 Proving a function is integrable

Simplifying the problem for a function f on an interval I:

- note/show that f is measurable
- replace f with $|f|$, then use 4.8.2

Solving the problem:

- if f and I are bounded, then f is integrable over I
- if I is unbounded, or f is unbounded on I, consider an increasing sequence of intervals (I_n) st. f is bounded on each I_n
  - apply the FTC, integration by parts, or substitution to solve $\int_{I_n} f$
  - then apply the Baby MCT
- use the Comparison test to find a simpler function that is easier to integrate/prove not integrable.

5 Convergence theorems

MCT v2 [5.1]: for a sequence of integrable functions (f_n) with

(1) $\forall n \; f_n \leq f_{n+1}$ a.e., and

(2) $\sup_n \int f_n < \infty$, then (f_n) converges a.e. to an integrable function f, and $\int f = \lim_{n\to\infty} \int f_n$

Proof: use 4.8 to ensure (1) is actually everywhere, and $f_n$is finite everywhere. Then apply the MCT v1 to $f_n - f_1$.

Fatou's Lemma [5.3]: for a sequence of non-negative measurable function (f_n),

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

Prove by applying the MCT to $g_r := \inf_{n\geq r} f_n$

Dominated Convergence Theorem (DCT) [5.4]: for a sequence of non-negative measurable function (f_n) st

1. $(f_n(x))$ converges a.e. to a limit $f(x)$
2. $\exists$an integrable function g st $\forall n \; |f_n(x)| \leq g(x)$ a.e

Then f is integrable, and $\int f = \lim_{n\to\infty} \int f_n$.

Proof: f is measurable, and integrable by comparison, then apply Fatou's lemma twice to show equality of integrals

Bounded Convergence Theorem (BCT) [5.6]: for a bounded interval I, if (f_n) is a sequence of functions integrable on I converging a.e. to f which is bounded by a constant c a.e. for all n.

Then f is integrable on I, and $\int_I f = \lim_{n\to\infty} \int_I f_n$

Proof by DCT.

Beppo Levi Theorem/ Lebesgue's Series Theorem [5.9] for a sequence (g_n) of integrable functions with $\sum_n \int |g_n| < \infty$, then $\sum_n g_n$ converges a.e. to an integrable function, and $\int \sum_n g_n = \sum \int g_n$. Prove by applying MCT for series to positive and negative parts of g_n

Alternate version of Beppo Levi [5.10] for a sequence (g_n) of integrable functions with $\sum_n |g_n|$ integrable, then $\sum_n g_n$converges a.e. to an integrable function, and $\int \sum_n g_n = \sum \int g_n$. Prove by applying Beppo Levi to $f_k = \sum_{n=1}^{k} g_n$.

6 Integrals depending on a parameter

Setup: given $f : \mathbb{R}^2 \to \mathbb{R}$, assuming $x \mapsto f(x,y)$ is integrable, we think about $F(y) = \int f(x,y)dx$

Continuous-parameter DCT [6.2]: given I, J are intervals in $\mathbb{R}$, $f : I \times J \to \mathbb{R}$ is a function with the following properties:

(1) $\forall y \in J$, $x \mapsto f(x,y)$ is integrable over I

(2) $\forall y \in J$, $\lim_{z \to y} f(x,z) = f(x,y)$ a.e. in $x \in I$ [continuous in y, the **outer parameter**, at almost all x]

(3) $\exists g : I \to \mathbb{R}$, an integrable function st $\forall y \in J$ $|f(x,y)| \leq g(x)$ at almost all x OR

(3') $\forall b \in J$, $\exists J_b \subseteq J$, an open sub-interval of J with $b \in J_b$, and $\exists g_b : I \to R$ integrable st $\forall y \in J_b$ $|f(x,y)| \leq g(x)$ at almost all x [implied by (3)]

Then, using one of condition (3) or (3'), $F(y) = \int_I f(x,y)dx$ is continuous on J (condition 1 ensures that is is integrable)

Proof: use the normal DCT on $f_n(x) = f(x, y_n)$ for any sequence $y_n \to y$ in J

Differentiating F [6.5]: given I, J are intervals in $\mathbb{R}$, $f : I \times J \to \mathbb{R}$ is a function with the following properties:

(1) $\forall y \in J$, $x \mapsto f(x,y)$ is integrable over I

(2) $\forall x \in I, y \in J$, $\frac{\partial f}{\partial y}(x,y)$ exists - the derivative in the *outer* parameter

(3) $\exists$ integrable $g : I \to \mathbb{R}$ st $\forall y \in J$ $|\frac{\partial f}{\partial y}(x,y)| \leq g(x)$ at almost all x OR

(3') $\forall b \in J$, $\exists J_b \subseteq J$, an open sub-interval of J with $b \in J_b$, and $\exists g_b : I \to R$ integrable st $\forall y \in J_b$ $|\frac{\partial f}{\partial y}(x,y)| \leq g(x)$ at almost all x [implied by (3)]

Then $F(y) = \int_I f(x,y)dx$ is differentiable on J, and $F'(y) = \int_I \frac{\partial f}{\partial y}(x,y)dx$

Proof: for any fixed $y \in J$, (y_n) a sequence converging to y with $y_n \neq y$, let $g_n(x) = \frac{f(x,y_n) - f(x,y)}{y_n - y}$, which is integrable over I, and converges to $\frac{\partial f}{\partial y}(x,y)$ as $n \to \infty$. The MVT says $\exists \xi \in [y_n, y]$ st $g_n(x) = \frac{\partial f}{\partial y}(x,\xi)$, so by (3) $|g_n(x)| \leq g(x)$ a.e.(x). Thus the DCT is applicable, so $\frac{F(y_n) - F(y)}{y_n - y} = \int_I g_n(x)dx \to \int_I \frac{\partial f}{\partial y}(x,y)dx$ as $n \to \infty$. Since the sequence was arbitrary, $\frac{F(y') - F(y)}{y' - y} = \int_I g_n(x)dx \to \int_I \frac{\partial f}{\partial y}(x,y)dx$ as $y' \to y$

7 Double integrals

for $f : \mathbb{R}^2 \to \mathbb{R}$, f can be integrable on $\mathbb{R}^2$, in which case its integral is$\int_{\mathbb{R}^2} f$. This is defined in the same way as integrability over $\mathbb{R}$, excepting the bits where we compare to the Riemann integral.

[Skipping 7.1 to just list the two as actually used]

Tonelli's theorem [7.3]: for $f : \mathbb{R}^2 \to \mathbb{R}$, a measurable function, if either these two is finite, then $f, |f|$ are integrable over $\mathbb{R}^2$, and so Fubini's theorem applies to f and $|f|$.

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| dx \right) dy, \qquad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| dy \right) dx$$

Fubini's theorem [7.2]: for $f : \mathbb{R}^2 \to \mathbb{R}$, integrable (i.e. over $\mathbb{R}^2$):

- $x \mapsto f(x,y)$ is integrable for almost all y,
- $F(y) = \int f(x,y)dx$ is integrable (for the y for which it is defined),
- $y \mapsto f(x,y)$ is integrable for almost all x,
- $G(x) = \int f(x,y)dy$ is integrable (for the x for which it is defined)

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)dx \right) dy = \int_{\mathbb{R}^2} f(x,y)d(x,y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)dy \right) dx$$

Changing variables [7.13]: Let $f : E \to \mathbb{R}$, for $E \subseteq \mathbb{R}^2$, and $T : E' \to E$ an injective differentiable function, and $E' \subseteq \mathbb{R}^2$ an open set.

f is integrable over $E \iff (f \circ T)|\det J_T|$ is integrable over E', where J_T is the Jacobian matrix of T.

So if $T : (u,v) \in E' \mapsto (x,y) \in E$

$$\frac{\partial(x,y)}{\partial(u,v)} := \det J_T = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial v}$$

So if either of $f, (f \circ T)|\det J_T|$ is integrable,

$$\int_E f(x,y)d(x,y) = \int_{E'} f(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| d(u,v)$$

Changing to polar coordinates [7.9]: Same setup as before, but specifically $E' = \{(r,\theta) : r \geq 0, \theta \in [0,2\pi], (r\cos\theta, r\sin\theta) \in E\}$, and $T(r,\theta) = (r\cos\theta, r\sin\theta)$, so $\frac{\partial(x,y)}{\partial(u,v)} = r$, so

$$\int_E f(x,y)d(x,y) = \int_{E'} f(r\cos\theta, r\sin\theta)r\, d(r,\theta)$$

8 L^p Spaces

Let $\mathcal{L}^p$ be the set of all measurable functions on $\mathbb{R}$ st $|f|^p$ is integrable, and $\mathcal{N} = \{f \in \mathcal{L}^p : f = 0$ a.e.$\}$, the equivalence class [0] under the relation $f \sim g \implies f = g$ a.e.

Let $L^p = \mathcal{L}^p/\mathcal{N}$, which is a vector space, as $(|f+g|)^p \leq (2\max(|f|,|g|)^p = 2^p \max(|f|^p, |g|^p) \leq 2^p(|f|^p + |g|^p)$.

Define $\|f\|_p = \left(\int |f|^p\right)^{1/p}$, which is a norm on L^p for $p \geq 1$. It clearly satisfies $\|f\|_p = 0 \iff f \in \mathcal{N}$, and $\|\alpha f\|_p = \alpha\|f\|_p$ for all $p \geq 0$

Only for $p \geq 1$ does $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, which is **Minkowski's Inequality [8.1]**

- if either f, g is in $\mathcal{N}$, then it is trivial
- so let $\alpha := \|f\|_p > 0, \beta := \|g\|_p > 0$
- $t \mapsto t^p$ is a convex and continuous function on $[0,\infty)$ [by looking at second derivative], so $(\lambda s + (1-\lambda)t)^p \leq \lambda s^p + (1-\lambda)t^p$
- Apply this with $\lambda = \frac{\alpha}{\alpha+\beta}$, $s = \frac{|f(x)|}{\alpha}, t = \frac{|g(x)|}{\beta}$, so $\left(\dfrac{|f|+|g|}{\alpha+\beta}\right)^p \leq \dfrac{1}{\alpha+\beta}\left(\dfrac{|f|^p}{\alpha^{p-1}} + \dfrac{|g|^p}{\beta^{p-1}}\right)$ for all x.
- We also have $\left(\dfrac{|f+g|}{\alpha+\beta}\right)^p \leq \dfrac{|f|^p}{\alpha^{p-1}} + \dfrac{|g|^p}{\beta^{p-1}}$ as $|f+g| \leq |f| + |g|$
- So by integrating we get $\dfrac{(\|f+g\|_p)^p}{(\alpha+\beta)^p} \leq \dfrac{1}{\alpha+\beta}\left(\dfrac{(\|f\|_p)^p}{\alpha^{p-1}} + \dfrac{(\|g\|_p)^p}{\beta^{p-1}}\right) = \dfrac{\alpha+\beta}{\alpha+\beta} = 1$
- And if we rearrange and take p-th roots, we get $\|f+g\|_p \leq (\alpha+\beta) = \|f\|_p + \|g\|_p$

Hölder's Inequality [8.2]: Let $p, q \in (1,\infty)$ with $1/p + 1/q = 1$, $f \in L^p, g \in L^q$. Then $fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p\|g\|_q$. Note if $p = q = 2$, this is the Cauchy-Schwartz inequality.

- $t \mapsto \log t$ is concave on $[0,\infty)$, because its second derivative $-t^{-2}$ is negative.
- So, $\frac{1}{p}\log s + \frac{1}{q}\log t \leq \log(\frac{s}{p} + \frac{t}{q})$.
- Exponentiating gives us that $s^{1/p}t^{1/q} \leq \frac{s}{p} + \frac{t}{q}$.
- Let $s := \left(\dfrac{|f|}{\|f\|_p}\right)^p, t := \left(\dfrac{|g|}{\|g\|_q}\right)^q$
- Thus, $\dfrac{|f|}{\|f\|_p}\dfrac{|g|}{\|g\|_p} \leq \dfrac{|f|^p}{p(\|f\|_p)^p} + \dfrac{|g|^q}{q(\|g\|_q)^q}$
- Integrating gives us $\dfrac{\|fg\|_1}{\|f\|_p\|g\|_p} \leq \dfrac{(\|f\|_p)^p}{p(\|f\|_p)^p} + \dfrac{(\|g\|_q)^q}{q(\|g\|_q)^q} = \dfrac{1}{p} + \dfrac{1}{q} = 1$
- So $\|fg\|_1 \leq \|f\|_p\|g\|_p$, so $fg \in L^1$.

Strict inclusion of L^p spaces in others [8.3]: For p_1, p_2 st $1 \leq p_1 < p_2 < \infty$ (strict inequality), if $f \in L^{p_2}(a,b)$, then:

- $f \in L^{p_1}(a,b)$
- $\|f\|_{p_1} \leq (b-a)^{1/p_1 - 1/p_2}\|f\|_{p_2}$

if $f_n \in L^{p_2}(a,b)$ and $\|f_n\|_{p_2} \to 0$, then $\|f_n\|_{p_1} \to 0$

Proof: apply Hölder's Inequality to $|f|^{p_1}$ and $\chi_{(a,b)}$ with $p = p_2/p_1, q = p_2/(p_2 - p_1)$, so $|f|^{p_1}\chi_{(a,b)} \in L_1$ and $\||f|^{p_1}\chi_{(a,b)}\|_1 = \int |f|^{p_1} \leq (\int |f|^{p_2})^{p_1/p_2}(b-a)^{(p_2-p_1)/p_2}$, and then take the p_1-th root, to get $\|f\|_{p_1} \leq (b-a)^{1/p_1 - 1/p_2}\|f\|_{p_2}$.

This only works for spaces of finite measure - e.g. not $(1, \infty)$

***L^p* is a complete measure space [8.5]:** for $p \in [1, \infty)$, (f_n) a Cauchy sequence in $\mathcal{L}^p$ - i.e. $\forall \varepsilon > 0 \; \exists N \; m, n \geq N \implies \|f_n - f_m\|_p < \varepsilon$. Then $\exists f \in \mathcal{L}^p$ st

1. a subsequence (f_{n_k}) of (f_n) exists st $\lim_{k \to \infty} f_{n_k}(x) = f(x)$ a.e.
2. $\lim_{n \to \infty} \|f_n - f\|_p = 0$

So L^p is a complete measure space.

Proof: see notes, since it's quite detailed

Egorov's Theorem [8.7]: Suppose that $f_n \to f$ a.e., and E is a measurable set with $m(E) < \infty$, and $\varepsilon > 0$. Then, there is a measurable subset $F \subseteq E$ with $m(E \backslash F) < \varepsilon$ st $f_n \to f$ uniformly on F - i.e., $\|f_n - f\|_{L^p(F)} \to 0$ for all $p \geq 1$

[No proof]

Sequence of step functions [8.8] if $f \in L^p(\mathbb{R})$ where $p \geq 1$, then there is a sequence of step functions ϕ_nst $\lim_{n \to \infty} \|f - \phi_n\|_p = 0$

8.1 Fourier transforms

Given $f \in \mathcal{L}^1(\mathbb{R})$, the Fourier transform of f is the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ as follows:

$$\widehat{f}(s) = \int_{\mathbb{R}} f(x)e^{-isx}dx$$

Properties of the Fourier transform [8.9]

1. $\forall s$ $|\widehat{f}(s)| \leq \|f\|_1$ because $|\widehat{f}(s)| = |\int_{\mathbb{R}} f(x)e^{-isx}dx| \leq \int_{\mathbb{R}} |f(x)e^{-isx}|dx = \int_{\mathbb{R}} |f(x)|dx = \|f\|_1$
2. $\widehat{f}$ is continuous by applying the continuous-parameter DCT [6.2] with $g(x) = |f(x)|$, which is integrable.
3. **Riemann-Lebesgue lemma:** $\widehat{f}(s) \to 0$ as $s \to \pm\infty$
   - for $f = \chi_{(a,b)}$, $\widehat{f}(s) = \frac{i}{s}(e^{-isb} - e^{-isa}) \to 0$ as $s \to \pm\infty$
   - thus it works for step functions, since the integral is linear.
   - for general $f \in \mathcal{L}^1(\mathbb{R})$, $\varepsilon > 0$, by 8.8 there is a step function φ st $\|f - \varphi\|_1 < \varepsilon$, and by the point above $\exists K > 0$ st $|\widehat{\varphi}(s)| < \varepsilon$ whenever $|s| > K$.
   - Then, $|\widehat{f}(s)| \leq |\widehat{f}(s) - \widehat{\varphi}(s)| + |\widehat{\varphi}(s)| \leq \|f - \varphi\|_1 + |\widehat{\varphi}(s)| \leq 2\varepsilon$ when $|s| > K$
4. given $g(x) = xf(x)$, $g \in \mathcal{L}^1(\mathbb{R})$, then $\widehat{f}$ is differentiable everywhere on $\mathbb{C}$, and $(\widehat{f})'(s) = -i\widehat{g}(s)$
   - by applying [6.5] with $|g|$ as the dominating function.
5. if f has cont. derivative $f' \in \mathcal{L}^1(\mathbb{R})$, then the Fourier transform of f' is $is\widehat{f}(s)$
   - by integrating by parts over intervals $[a_n, b_n]$ with $a_n \to -\infty, b_n \to \infty$, so $f(a_n) \to 0, f(b_n) \to 0$

9 Absolutely Continuous Functions

we want a class $\mathcal{A}$ of functions on an interval $[a, b]$ st

1. if $F \in \mathcal{A}$ then F is differentiable a.e., $F' \in L^1(a, b)$, and $\int_a^x F'(y)dy = F(x) - F(a)$ for all $x \in [a, b]$
2. if $f \in L^1(a, b)$ and $F(x) := \int_a^x f(y)dy$ for $x \in [a, b]$, then $F \in \mathcal{A}$ and $F' = f$ a.e.

Thus, this class of functions satisfies the FTC "in both directions", and is the class of functions of the form $F(x) := c + \int_a^x f$ for some $f \in L^1(a, b), c \in \mathbb{R}$.

It turns out, by two theorems that go unproved, that $\mathcal{A}$ is also equivalent to the set of functions that are absolutely continuous on $[a, b]$:

A function $F : I \to \mathbb{R}$ is **absolutely continuous** on an interval I if $\forall\varepsilon > 0, \exists\delta > 0$ st

($\forall n \in \mathbb{N}$, disjoint subintervals of I (a_r, b_r) for $r \in \{1...n\}$ with $\sum_{r=1}^n (b_r - a_r) < \delta$) $\implies$

$$\sum_{r=1}^{n} |F(b_r) - F(a_r)| < \varepsilon$$

10 Integrable functions

$x^{-a}, (1, \infty)$ for $a > 1$ by <u>Baby MCT</u>

$x^a, (0, 1)$ for $a > -1$ by Baby MCT

$f(x) = \begin{cases} x & x \,\text{rational} \\ 0 & x \,\text{irrational} \end{cases}$, $\mathbb{R}$ is measurable, and so has integral 0

$\sin(1/x)$,$(0, 1]$ as both are bounded

$x^n e^{-x}$,$[0, \infty)$ for $n \in \mathbb{N}$ (i.e. incl e^{-x}) since $\exists a_n \in \mathbb{R}$ st $\forall x \geq a_n$ $e^{x/2} \geq x^n$, so on $(0, a_n)$ is is bounded on bounded, and on $[a_n, \infty)$, compare to $e^{-x/2}$, and use Baby MCT on that.

$1/\sqrt{x}, (0, 1]$ by <u>Baby MCT</u>

$(\log x)e^{-x}, (0, \infty)$ by splitting to $(0, 1]$ and $[1, \infty)$, comparing to $1/\sqrt{x}$ and xe^{-x} resp.

$x^a \ln x, (1, \infty)$ for $a < -2$ by comparison to x^{a+1}, and for $a \in [-2, 1)$ by the Baby MCT manually

$\sqrt{\csc x}, (0, \pi)$ by splitting to $[0, 1]$, $[1, \pi - 1]$, $[\pi - 1, \pi]$ on which it is bdd by $\sqrt{2/x}$, bounded on a bounded interval, and then by symmetry to $[0, 1]$

$e^{-x} \sin x, (0, \infty)$ by the DCT: integrate by parts on $(0, n\pi)$, and find the limit of those (bound provided by e^{-x})

$x^k \log^2 x, (0, 1)$ or $e^{ku}u^2, (1, e)$ by integration by parts

$e^{-x^2}, \mathbb{R}$ by comparison to e^{-x} for $x \geq 1$, boundedness for $[0, 1]$ and symmetry for $x < 0$, or indeed by bounding by $2/(2 + x^2)$

$e^{-x^2}/\sqrt{x}, [0, 1]$ by taking its Taylor series and integrating terms, then using Beppo Levi

$\frac{x^2}{e^{x^2}-1}, (0, \infty)$ or $e^{-x^2/2}, (0, \infty)$ since it is $\leq 2/(2 + x^2)$ by looking at the Taylor series

$2/(2 + x^2), (0, \infty)$ by the MCT and the arctan substitution.

11 Non-integrable functions

$\tan x, (0, \pi/2)$ by the Baby MCT cause the integral on $(0, \pi/2 - 1/n)$ tends to ∞

$\begin{cases} -1^n/n & x \in [n, n + 1) \\ 0 & x < 1 \end{cases}$ cause positive and negative parts have infinite integrals

$x^a \log x, [1, \infty)$ for $a \geq 0$ by comparison to $\log x$, and for $a \in [-1, 0]$ by the Baby MCT

$\log x, [1, \infty)$ by the Baby MCT

$\sin x/x$, $\cos x(1 + x)$ or similar by 4.10.5 - i.e. integrate over π-long regions and show the sum over those is infinite

$\sin 1/x, [1, \infty)$ by sheet 2 Q10 part x

12 Useful integrals

[Not proofs of integrability]

$\int \frac{f'}{f} = \ln|f(x)| + C$

$\sin x \geq \dfrac{2}{\pi} x$ on $[-\pi, \pi]$