Candidate Number:

inner prod spce: $\langle\,\rangle$ bi/sesq; linear, (skew) symmetric, pos. $\|x\| := \sqrt{\langle x,x\rangle}$

Cauchy-Schwartz: $|\langle x,y\rangle| \le \|x\|\,\|y\|$ $x := \alpha x$ st $\alpha\langle x,y\rangle \in \mathbb{R}$, $\|x+t y\|^2 \ge 0$, discriminant

Parallelogram: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \;\forall x,y$ $+\,\&$ for $t=\pm1$

ortho: nb: $\|\cdot\|$ sat. ▱ law $\Rightarrow$ $\|\cdot\|$ from $\langle,\rangle := \frac14(\|x+y\|^2 - \|x-y\|^2)$, in $\mathbb{C}$ $+i\times$ same w/ $y = iy$

$Y \subseteq X$: $Y^\perp$ closed in X, $Y \subset Y^{\perp\perp}$, $Y^\perp = (\overline{\operatorname{span} Y})^\perp$ closed: $\langle\rangle$ cont, $\subset Y^{\perp\perp}$ def, $= (\operatorname{span} Y)^\perp$: $\supseteq\checkmark$, $\le$ by cont, lin of $\langle\rangle$

closest pont in closed, convex $X \subseteq$ Hilb. to $y \notin X$ exists, is unique $d_n \to d$, $d_n = \|x - y_n\|$, ▱ to $\frac12(x-y_n)$, $\frac12(x-y_m)$ $\Rightarrow$ (y_n) Cauchy, as $d \le \|x - \frac12(y_n+y_m)\| \Rightarrow$ conv $\Rightarrow \exists\checkmark$ | uniq: ▱ again on $\frac12(x-y)$, $\frac12(x-y')$, show $=$

proj. theorem: Y closed subspace in X Hilb $\Rightarrow X = Y \oplus Y^\perp$ $x \in X$, $y_0 :=$ closest to x in Y $\|x-y_0\|^2 \le \|x-(y_0-ty)\|^2 = \|x-y_0\|^2 + 2t\operatorname{Re}\langle x-y_0, y\rangle + t^2\|y\|^2 \Rightarrow [\operatorname{Re}]\ \langle x-y_0, y\rangle = 0$ as $\forall t$

closed linear span: smallest clsd lin subsp of X cont. S $= S^{\perp\perp}$. as $(\overline{\operatorname{span} S})^\perp = S^\perp$

orthonormal basis: orthonormal set S st $\operatorname{clspan}(S) = \overline{\operatorname{span} S} = X$

every Hilb space has an orthonormal basis P only separable, Gram-Schmidt.

Pythagorean theorem: $\|x\|^2 = \sum_1^m |\langle x,x_n\rangle|^2 + \|x - \sum_1^m \langle x,x_n\rangle x\|^2$] on Hilb calc w/ $\langle\rangle$

Bessel: $\|x\|^2 \ge \sum_1^\infty |\langle x,x_n\rangle|^2$ $\longleftarrow$ for $\{x_1, \dots x_m\}$ orthonormal] use ↖ in limit

X Hilb, $S = \{x_1, \dots\}$ orthonorm. seq. $\operatorname{cl}(\operatorname{span}(S)) = \{\sum^\infty a_n x_n : (a_n) \in \ell^2\}$, $\sum a_n x_n$ conv as ℓ^2

Parseval's: $\|x\|^2 = \sum_1^\infty |a_n|^2$ $\tilde{x} := \sum$, $x - \tilde{x}$ perp to all x_n, $\in Y^\perp = \{0\}$ so $\checkmark$ Parsevals: Pyth in lim

Hilbert ex: $\mathbb{C}^n$ or $\mathbb{R}^n$ w/ $\langle\rangle = \sum_1^n x_k \bar{y}_k$, ℓ^2 w/ $\sum^\infty x_k \bar{y}_k$

closed subsp. of Hilb., Bergman: $A^2(D) = f \in L^2$, holo on D

H only: $H^2(\mathbb{T})$: $f \in L^2(-\pi,\pi)$ st $\hat{f}(n) = \sum_{n\ge0} a_n e^{inx}$: clsd subsp of $L^2(-\pi,\pi)$

Sobolev: $H^1(a,b) \subseteq L^2(a,b)$ st $u = A + \int_a^x v(t)\,dt$ $v \in L^2(a,b)$ a.e. $\langle u,v\rangle = \int_a^b u\bar{v} + u'\bar{v}'$

$\langle Ax,y\rangle = \langle Bx,y\rangle \;\forall x,y \Rightarrow A = B$

Riesz-rep: $\boxed{X \text{ Hilb}}$, $f: X \to \mathbb{F}$ bdd lin func: $\exists x'$ st $\forall x$ $f(x) = \langle x, x'\rangle$.

x unique, + $\|x\| = \|f\|_\infty$. P: $Y := \ker f$, take $y^\perp \in Y^\perp$, $x' := \overline{f(y^\perp)}/\|y^\perp\|^2\, y^\perp$.

Adjoint operators: $A^*: Y \to X$ def by $y \mapsto y'$, where $\langle Ax,y\rangle = \langle x,y'\rangle$

$A \in B(X,Y)$ [both Hilb] [y' exists by R-r on $x \mapsto \langle Ax,y\rangle$]

1) A^* is unique, a BLO 2) $\langle Ax,y\rangle_Y = \langle x,A^*y\rangle_X$ 3) $\|A\| = \|A^*\|$ 4) $A^{**} = A$

5) $(aA+bB)^* = \bar{a}A^* + \bar{b}B^*$ 6) $(ST)^* = T^*S^*$ 7) $I^* = I$ 8) A invertible $\Leftrightarrow A^*$ inv.

self-adjoint: $T^* = T^*$ | $\|T\|_{x \to x} = \sup\{|\langle Tx,y\rangle| : \|x\|=\|y\|=1\} = \sup\{|\langle Tx,x\rangle| : \|x\|=1\}$ if self-adj.

$\|A^*A\| = \|A\|^2 \Rightarrow \|A^2\| = \|A\|^2$ if self-adj. | $\ker A = (\operatorname{Im} A^*)^\perp$, $(\ker A)^\perp = \overline{\operatorname{Im} A^*}$

$X = Y \oplus Z$, both clsd. subsp. $P(y+z) = y$, TFAE: 1) $Z = Y^\perp$ 2) $P^* = P$ 3) $\|P\| \le 1$ [$\|P\| = 1/0$]

Unitary: linear operator between 2 Hilbs i) isometric ii) surjective

n isometry + $T(0)=0 \Rightarrow$ real linear. Hilb

isometric $\Leftrightarrow \forall x,y$ $\langle Tx,Ty\rangle = \langle x,y\rangle \Leftrightarrow T^*T = I_X$

unitary $\Leftrightarrow U^*U = I_X$, $UU^* = I_Y \Leftrightarrow$ both U, U^* isometric

R-N: $\nu \ll m$: $d\nu = g\,dm$, for $g \ge 0$, int wrt m & $\nu(E) = \int_E g\,dm$

Pf: $X := L^2(A, \nu+m)$ by R-r $\int_A f\,dm = \int_A fh\,d(\nu+m)$ $\Rightarrow g = \frac{1-h}{h}$, prove $0 < h \le 1$ except on null.

Adjoint proofs

1) uniq by R-r, lin $\checkmark$ bdd: $\|A^*x\|^2 \overset{CS}{\le} \|A\|\,\|A^*x\|\,\|x\|$

3) $\|Ax\|^2 = \langle Ax,Ax\rangle = \langle x, A^*Ax\rangle$ then repeat w/ A^*

$\le \|x\|\,\|A^*A\|\,\|x\|$ by C-S, A^*A BLO

4) $\langle Ax,y\rangle = \langle A^*y, x\rangle = \langle A^{**}x, y\rangle$

5 ✓ 6) ✓ 7) ✓ 8) $(A^{-1})^*$ exists

$I_X = I_X^* = (AA^{-1})^* = (A^{-1})^*A^*$ + reverse

$\leftarrow$ $K = \sup_x$ $\le \|T\|$ $\forall \epsilon\, \exists x,y$ $|\langle Tx,y\rangle| > \|T\| - \epsilon$, WLOG Real

$4(\|T\| - \epsilon) \le 4\operatorname{Re} = \langle T(x+y),(x+y)\rangle$.. ▱ $= 4K$ $\in (\operatorname{Im} A^*)^\perp$, $y \in \operatorname{Im} A^*$

$\ker \subseteq$: $x \in \ker$, $y \in \operatorname{Im} A^*$: $\langle x,y\rangle = \langle Ax,y\rangle = 0$. $\supseteq$: $\langle x,y\rangle = 0 \Rightarrow \langle Ax,w\rangle = 0$

take $w = Ax$. $(\ker A)^\perp = (\operatorname{Im} A^*)^{\perp\perp} = (\overline{\operatorname{span} \operatorname{Im} A^*})^{\perp\perp}$ [or L5]

as A^* lin. $\Rightarrow = (\overline{\operatorname{Im} A^*})^{\perp\perp} = \overline{\operatorname{Im} A^*}$ as closed

$1 \Rightarrow 2$: decompose x by $2 \Rightarrow 3$ $\|P\| = \|P\|^2$

$3 \Rightarrow 1)$ $P = 0$ ✓ $\langle y,x\rangle \le \langle y+2, y+2\rangle$, $\langle y,2\rangle = 0$ etc take y st $\le$ ✗

by considering $m(F)$ $F = \{h \le 0\}$ OR $\{h > 1\}$

nb $\int_A (1-h)\,dm = \int_A gh\,d\nu$ $\forall$

S dense in M if $\bar{S} = M$, nowhere dense in M if $\operatorname{int}(\bar{S}) = \emptyset$.

Baire category theorem: a non-empty complete metric space is never the union of countably many nowhere dense sets

pf: ✗, construct Cauchy seq $\notin \bigcup A_n$, by $\bar{B}(x_n,r_n) \subseteq B(x_{n-1}, r_{n-1}) \setminus A_n$, non-empty & open, $r_n < 1/2^{n-1}$

Uniform Boundedness X Banach, Y normed, $\mathcal{F} \subset B(X,Y)$ ($\forall x$ $\{\|Tx\|_Y : T \in \mathcal{F}\}$ bdd) $\Rightarrow \{\|T\| : T \in \mathcal{F}\}$ bdd.

pf: $A_n := \{x : \|Tx\| \le n \;\forall T \in \mathcal{F}\}$ $\leftarrow$ is closed. by Q, $\bigcup A_n = X$, so at least 1 not nowhere dense

$\Rightarrow \exists n_0, x_0, r_0$ st $B(x_0, r_0) \in A_{n_0}$. $\|z\| < r_0 \Rightarrow x_0 + z \in B \Rightarrow$.. $\|Tz\| \le \|T(z+x_0)\| + \|Tx_0\| \le 2n_0$ so BLO.

consequences: 1) X Hilb, $\sup_{T \in \mathcal{F}} |\langle Tx,y\rangle| < \infty$ $\forall x,y$ $\Rightarrow \{\|T\| : T \in \mathcal{F}\}$ bdd pf: $K_{Tx} \in X^*$ defs $\langle \cdot, Tx\rangle$

2) $(T_n)_{n\ge1} \in B(X,Y)$. TFAE: a) $\exists T \in B(X,Y)$ $\forall x$ $T_n x \to Tx$ b) $\forall x$ $T_n x$ conv $y \mapsto \langle y, Tx\rangle$ PUB twice.

c) $\exists M$, dense $Z \subseteq X$ st $\|T_n\| < M \;\forall n$, $(T_n z)$ conv $\forall z \in Z$ pf: $a \Rightarrow b$ ✓, $b \Rightarrow c$ PUB, $c \Rightarrow a$) $\forall x$ $T_n x$ Cauchy + BLO.

Open mapping: $T \in B(X,Y)$ surj, X,Y Banach: T is an open map [i.e. $T(U)$ open if U open] p: only $B(0,r)$, for 1 r, as $T(B(0,r)) = rT(B(0,1))$, and $T(U) = Tx + T(-x+U)$.

$Y = \bigcup_n T(B(0,n))$. Baire $\Rightarrow \exists n$ $\overline{T(B(0,n_0))}$ has non-empty interior $\Rightarrow$ contains $B_Y(y_0, r_0) \Rightarrow$ cont $B(0,r_0)$ as $\overline{T(B_X(0,n_0))}$ is symmetric & convex. show $\overline{T(B_X(0,n_0))} \subseteq T(B_X(0,3n_0))$: $y \in \overline{T(\cdot\, n_0)}$: $x_k \in B_X(0, 2^{1-k} n_0)$

Inverse mapping a bijective BLO on 2x Banach has a bdd inverse: ↑ means inverse is cont. st $\|y - \sum_j^k Tx_j\|_Y \le 1/2^k\, r_0$ by ind, as $y - \sum^{k-1} Tx_j \in B_X(0, 2^{-k} n_0)$. nb. using closure of Im of T.

$T \in B(X,Y)$ 2x Hilb: TX closed $\Leftrightarrow T^*Y$ closed. $x = \sum x_k$ exists as $|x_k| \le 2^{1-k} n_0$, $y = Tx \Rightarrow y \in T(B_X(0, 3n_0))$.

Closed graph: T linear $X \to Y$ 2x Banach, bdd $\Leftrightarrow \Gamma(T) = \{x,y : y = Tx\}$ closed in $X \times Y$ pf: $\Leftarrow \Gamma(T)$ a Banach spce, proj maps $P_1 \to X$, $P_2 \to Y$ $T = P_2 \circ P_1^{-1}$.

direct sum projection is bdd on Banach spaces pf: use graph

L45 prof: only need $T^*Y \Rightarrow TX$, as $T^{**} = T$. $W := T^*Y$ closed.

$Z := \overline{TX} \subseteq Y$. $S \in B(X,Z)$ def by $Sx = Tx$ $\forall x \in X$. $S^*: Z \to X$

$Z = \overline{\operatorname{Im} S} = (\ker S^*)^\perp \Rightarrow \ker S^* = \{0\}$ as $Z^\perp = \{0\}$. so S^* injective.

$\operatorname{Im} S^* = W$: $P =$ ortho proj $Y \to Z$: $\forall x,y$

$\langle Tx,y\rangle_Y = \langle Sx, Py\rangle_Y$ as $Sx, y - Py\rangle_Y = \langle x, S^*Py\rangle \Rightarrow T^* = S^* \circ P$

so S^* bdd, bij $Z \to W$. $V := S^*$, $V \in B(Z,W)$.

V has bdd inv $V^{-1} \in B(W,Z) \Rightarrow V^*$ invertible. $T\,\sigma(V^*)^{-1} = I_Z$:

$\langle Tw, z\rangle_Y = \langle Sw, z\rangle_Z = \langle w, S^*z\rangle = \langle w, Vz\rangle = \langle V^*w, z\rangle$ where $w := (V^*)^{-1}z$

✓ $\forall z$ so $Tw = z$. Thus $Z \subseteq TX \Rightarrow \overline{TX} = Z = TX$.

$z = T((V^*)^{-1}z)$, $(V^*)^{-1}z \in X$.

Weak convergence $x_n \rightharpoonup x \Leftrightarrow \forall f \in X^*$ $f(x_n) \to f(x)$. in fin dim, $x_n \rightharpoonup x \Leftrightarrow x_n \to x$.

Hilb: $x_n \rightharpoonup x$, $\|x_n\| \to \|x\| \Rightarrow x_n \to x$. p: $\rightarrow$ · | orthonormal seq (in Hilb) $\rightharpoonup 0$ [R-r, Bessel] but not $\to 0$. as not conv in norm

$x_n \rightharpoonup x$ in normed space: $\exists M$ $\|x_n\| \le M \;\forall n$. $T_n \in X^{**}$: $\ell \mapsto \ell(x_n)$, PUB as $\ell(x_n)$ conv + bdd. | Schur: $a^{(n)} \rightharpoonup a$ in $\ell^1 \Rightarrow a^{(n)} \to a$. too difficult

$\hookrightarrow \|x\| \le \liminf \|x_n\|$: ($\exists$ st $\ell(x) = \|x\|$, $\|\ell\|_* = 1$. Mazur: K closed + convex in normed space, $(x_n) \in K$, $x_n \rightharpoonup x \Rightarrow x \in K$. $0 \ne \ell_0, c$: $\ell_0(y) \le c \le \ell_0(z)$ $\forall y \in K$, $z \in B(x,r)$ $\forall w$ but ✗ to $\ell_0 \ne 0$

[$\Rightarrow \ell_0(x_n), \ell_0(x) \le c$ by weak conv $\Rightarrow \ell_0(x) \le \ell_0(z)$

[$\forall z \in B(x,r) \Rightarrow \ell_0(w) = \frac{1}{r}(\ell_0(x+rw) - \ell_0(z)) \ge 0$

Weak sequential compactness: $A \subseteq X$ Banach if $\forall (x_n) \subset A$ $\exists$ subseq $\rightharpoonup y \in A$ | pf. for X real: # $x \notin K \Rightarrow \exists$ hyper sep $B(x,r)$, K. $\ell_0(w) = \ell_0(x+w) - \ell_0(x) \le c - \ell_0(x)$: $w \in B(0,r) \Rightarrow \ell_0$ bdd

in Banach X, X reflexive $\Leftrightarrow \bar{B}(0,1)$ is w-seq-c & every bdd seq has a w-conv subseq pf: only $\Rightarrow$ for Hilb: diag process w/ $\langle x_n, x_1\rangle$

bdd $\Rightarrow$ BbB W: conv subseq $\to \forall m \le k$ $\langle x_{n_j}^{(k)}, x_m\rangle$ conv as $k \to \infty$, $x_{n_j} = x_{n_j}^{(j)}$. $Y =$ span x_n's. $X = \bar{Y} \oplus Y^\perp$, $\langle x_{n_j}, x\rangle$ conv (only) $\forall x$

def $\ell(x) = \lim_j \langle x, x_{n_j}\rangle$ a BLO. by R-r $\exists x'$ $\ell(x) = \langle x, x'\rangle$ $\forall x \in X$ by R-r [note flip $\langle\cdot\rangle$ w/ 2x conjugate] $x_{n_j} \rightharpoonup x'$

$K \ne \emptyset$, K closed, convex $\subseteq X$ refl. Banach $\forall x \in X$ $\exists$ (non-uniq?) $y \in K$ st no other point in K is closer.

refl $\Rightarrow \bar{B}(0,1)$ w-sc: $d_n \to d(x,K)$, $d_n = \|x - y_n\|$ $\|y_n\| \le \|y_n - x\| + \|x\|$ so bdd seq, w-conv subs, w-lim ✓

L50: $\|x - x_n\|^2 = \|x\|^2 + \|x_n\|^2 - \langle x,x_n\rangle - \langle x_n,x\rangle \to 0$. ↓ both lin funcs

extended Hyper plane sep theorem

A, B convex in normed X, $A \cap B = \emptyset$, 1 has interior point

A, B sep by hyp. plane. $\exists$ $\ell \ne 0 \in X^*$, c st

$\operatorname{Re} \ell(x) \le c \le \operatorname{Re} \ell(y)$ $\forall x \in A, y \in B$

non-extended: take $B = \{y\}$, $c = \operatorname{Re} \ell(y)$

Candidate Number:

Fourier series

some orthogonal bases: on $L^2(-\pi,\pi)$, the trigonometric functions $\{1/\sqrt{2\pi}, 1/\sqrt{\pi}\sin nx, 1/\sqrt{\pi}\cos nx\}$ or $\{1/\sqrt{2\pi}e^{inx}, n\in\mathbb{Z}\}$

on $L^2(-1,1)$ Legendre polys $p_n(t)$ st $\forall m\neq n$ $\int_{-1}^1 p_m(t)p_n(t)=0$, also coeff in t^n of $1/\sqrt{1-2xt+t^2}$: $1, x, \frac{1}{2}(3x^2-1)$

$L^2((0,\infty); e^{-t}dt)$ Laguerre "$L_n(t)$" sols to $xy''+(1-x)y'+ny=0$, otho wrt $\int_0^\infty fge^{-x}dx$: $1, 1-x, \frac12(x^2-4x+2)$

$f(x)\sim\mathcal{F}(f)=\sum_{n=-\infty}^{\infty}a_ne^{inx}$ \\ $\sum_{n=-\infty}^{\infty}\langle f,e_n\rangle e_n$ when $f\in L^2(-\pi,\pi)$, $\langle\cdot,\cdot\rangle=\int f\bar g$, $e_n:=e^{inx}/\sqrt{2\pi}$

$a_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx$

termwise diff & integ: given $f\in L^1_{loc}(\mathbb{R})$ / $L^1(-\pi,\pi)$, $F(x)=\int_a^x f(t)dt$. F 2π-ple, $F\sim\sum_{\mathbb{Z}}c_ne^{inx}\Rightarrow f\sim\sum inc_ne^{inx}$; f 2π-ple, $f\sim\sum c_ne^{inx}\Rightarrow F(x)-c_0x$ 2π-pd & $F(x)-c_0x\sim C_0+\sum_{n\neq0}\frac{c_n}{in}e^{inx}$.

partial sums: $S_Nf(x)=\sum_{-N}^{N}a_ne^{inx}=\int_{-\pi}^{\pi}f(t)k_N(x-t)dt$ $=\int_0^\pi(f(x+t)+f(x-t))k_N(t)dt$

$k_N(x)=\frac{1}{2\pi}\sum_{-N}^{N}e^{inx}=\frac{1}{2\pi}\frac{\sin((N+1/2)x)}{\sin x/2}$

divergence: $\exists 2\pi$ ple cont func f st $\mathcal{F}(f)$ diverges at 1 point. pf: $\cancel{X}$ means $A_N: f\mapsto S_Nf$ bdd, but $\|A_N\|=\int|k_N|\geq CLnN$.

$(0,1]\ni\alpha$ - Hölder cont at x if $\exists A>0,\delta>0$ st $\forall|h|<\delta$ $|f(x+h)-f(x)|\leq A|h|^\alpha$. $\alpha=1$ = Lipschitz

comp. letenes: $f\in L^2(-\pi,\pi)$, 2π-ple $\lim_N S_Nf\,[=\mathcal{F}(f)]=f$. i.e. $\{\frac{1}{\sqrt{2\pi}}e^{inx}\}$ an orthonormal basis of $L^2(-\pi,\pi)$ ⊕ all coeffs $=0\Rightarrow f=0$ a.e.

$f\in L^1(-\pi,\pi)$, 2π ple, α-Hö at $x_0\Rightarrow\lim_N S_Nf(x_0)=f(x_0)$ pf: WLOG $f(x_0)=0$, non-const, use $S_N=\int_0^\pi\cdots k_N\,dt$ split integral to $\int_\delta^\pi\leftarrow$ use R-L, $\int_0^\delta$ use Hö-cont $\leq 2A\int_0^\delta|t|^\alpha|k_N(t)|\leq\frac{A}{\sqrt{\pi}}\int\frac{H|t|^\alpha}{\sin t/2}\leq\frac{A}{\alpha}\delta^\alpha$ $[\sin t/2\geq t/\pi]$

Riemann-Lebesgue: $f\in L^1(-\pi,\pi)$ as $k\to\infty$, $\int_{-\pi}^{\pi}f(t)e^{ikt}dt\to0$ pf: Bessel if $f\in L^2$, $f=g+h$, $g\in C[-\pi,\pi]\subseteq L^2(-\pi,\pi)$, $\|h\|_{L^1}\leq\varepsilon$, $|\int\cdots h\cdots|\leq\varepsilon$.

$\|k_N\|\to\infty$ as $N\to\infty$, consider $x=2$

if differentiable, then coeffs d_n of f' are $n c_n$.

Spectral Theory

X Banach (i.e. complete), $T\in B(X)$

$\sigma(T)=\{\lambda:\lambda I-T$ not inv$\}$, $\rho(T)=\mathbb{C}\setminus\sigma(T)$, $R_\lambda(T)=(\lambda I-T)^{-1}$ // resolvent

point sp: $\sigma_p(T):=\{\lambda:\ker(\lambda I-T)\neq\{0\}\}\leftarrow$ eigenvalues, non triv $\in\ker$ = evects

residual sp: $\sigma_r(T):=\{\lambda:\ker(\lambda I-T)=\{0\}, Im(\lambda I-T)$ not dense in $X\}$

continuous sp: $\sigma_c(T)=\{\lambda:\ker=\{0\}, Im(\lambda I-T)$ proper dense $\subseteq X\}$

approx psp: $\sigma_{AP}(T)=\{\lambda:\exists x_n\in X$ $\|x_n\|=1$, $\|Tx_n-\lambda x_n\|\to0\}$

Properties

$\sigma_p\subseteq\sigma_{AP}$, $\sigma=$ disjoint $\sigma_p\cup\sigma_r\cup\sigma_c$
$=\sigma_{AP}\cup\sigma_r$, but not disjoint

σ: closed, non-empty (FA1)

$\lambda\in\sigma\Rightarrow|\lambda|\leq\|T\|$, $|\lambda^k|\leq\|T^k\|$ & $\sigma_r(T)\subseteq\sigma_{ap}(T')$

$\sigma(T)=\sigma_{AP}(T)\cup\sigma_p(T')$ no prf

$\sigma_c(T)\subset\sigma_{AP}(T)$: if $\lambda\notin\sigma_{AP}$, $\exists c$ $\|\cdot\|\geq c\|x\|$ $\forall x$ $\Rightarrow$ inv of $\lambda I-T$ on $Y=Im(\lambda I-T)$ is BLO, ext. to all X as Y dense $\Rightarrow$ Y not proper for $\cancel{X}$.

prf if Hilb: $p\in X\setminus Y$, x_n st $(\lambda I-T)x_n\to p$, weak seq comp $x_n\rightharpoonup x$ if bdd $\Rightarrow p=(\lambda I-T)x$ so x_n unbdd $z_n:=x_n/\|x_n\|$ ✓ for $\lambda\in\sigma_{AP}$.

U unitary $\Rightarrow\forall\lambda\in\sigma(U)$ $|\lambda|=1$ $\|U\|=1$, $U^{-1}=U^*$

$|\lambda|<1\in\sigma$: $\bar\lambda I-U^*$ not inv $\Rightarrow\bar\lambda U-I=U(\bar\lambda I-U^*)$ not inv
$\Rightarrow 1/\bar\lambda\in\sigma(U)$ but $|1/\bar\lambda|>1$

if X Hilbert on $\mathbb{C}$:

Ⓐ proof by linearity of $\langle\rangle$
Ⓑ proof: $U:=(\lambda I-T)^{-1}$: U^* ✓for $(\bar\lambda I-T^*)$ using $\forall x,y$ $\langle x,\cdot y\rangle=\langle\cdot x,y\rangle$
Ⓒ ✓by Ⓐ, PIL23. ✓same for Ⓓ

Ⓐ $(\lambda I-T)^*=\bar\lambda I-T^*$

Ⓑ $\lambda I-T$ inv $\Leftrightarrow(\lambda I-T)^*$ inv $\Rightarrow\lambda\in\sigma(T)\Leftrightarrow\bar\lambda\in\sigma(T^*)$

Ⓒ $\ker(\lambda I-T)=Im(\bar\lambda I-T^*)^\perp$

Ⓓ $\ker(\lambda I-T)^\perp=\overline{Im(\bar\lambda I-T^*)}$

T normal $(T^*T=TT^*)\Rightarrow\ker(\lambda I-T)=\ker(\bar\lambda I-T^*)$, $S:=\lambda I-T$ normal $\Rightarrow\|Sx\|^2=\langle x,S^*Sx\rangle=\langle x,SS^*x\rangle=\langle S^*x,S^*x\rangle=\|S^*x\|$

T self-adjoint $\Rightarrow\sigma_p(T)\subset\mathbb{R}$ $\lambda\|x\|^2=\langle Tx,x\rangle=\langle x,Tx\rangle=\bar\lambda\|x\|^2$

$\sigma(T)=\sigma_{AP}(T)\cup\{\bar\lambda: \lambda\in\sigma_p(T^*)\}$ $\supset$ ✓by ↑ $\subseteq$: $\lambda\in\sigma(T)\setminus\sigma_{AP}(T)$: by Ⓒ $\ker(\bar\lambda I-T^*)\neq\{0\}\Rightarrow\bar\lambda\in\sigma_p(T^*)$ nb all real, so can simp

if T self-adjoint:

① $\sigma(T)\subset\mathbb{R}$ have $\sigma_p\subseteq\mathbb{R}$: $\sigma_{AP}(T^*=T)$: λ, x_n: $\|x_n\|=1$, $\|(\lambda I-T)x_n\|\to0$: $\lambda x_n-\langle Tx_n,x_n\rangle=\lambda I-Tx_n\ldots$

② $\sigma(T)=\sigma_{AP}(T)=\sigma_p\cup\sigma_c(T)$ $\leq|x\|(\lambda I-T)x_n\|\to0$. $\langle Tx_n,x_n\rangle=\overline{\langle Tx_n,x_n\rangle}$ as self adj so $\lambda=$ lim of real vals

③ e-vects for different e-vals are ortho

② $\lambda\in\sigma_R(T)\Rightarrow\ker(\lambda I-T)=0$, Im not dense $\Rightarrow(\bar\lambda I-T^*)\neq\{0\}\Rightarrow\bar\lambda\in\sigma_p(T^*)\Rightarrow\lambda=\bar\lambda$ $T=T$ $\cancel{X}$

③ $\langle Tv_1,v_2\rangle=\langle v_1,Tv_2\rangle\Rightarrow(\lambda_1-\bar\lambda_2)\langle v_1,v_2\rangle=0\Rightarrow\langle v_1,v_2\rangle=0$ as $\lambda_1,\lambda_2\in\mathbb{R}$, $\neq$.

④ $r(T):=rad(\sigma(T))=\|T\|$ $n=2^k$: $\|T^n\|=\|T\|^n$, $r(T)=\lim\|T^n\|^{1/n}$ from FA1

⑤ $\sigma(T)\subseteq[a,b]$, $a,b:=\inf/\sup_{\|x\|=1}\langle x,Tx\rangle$ $\sigma=\sigma_{AP}$, $\lambda\in\sigma_{AP}$
$\lambda x_n-Tx_n\to0$, C-S $\Rightarrow\lambda-\langle Tx_n,x_n\rangle\to0\Rightarrow\lambda\in[a,b]$

$a,b\in\sigma(T)$: $|a|,|b|\leq r(T)=\|T\|\leq\max\{|a|,|b|\}\Rightarrow$ 1 of $a,b\in\sigma$ else radius smaller

$c\in\mathbb{R}$: $\sigma(cI+T)=c+\sigma(T)$, $a'=a+c$, $b'=b+c$, so both $\in\sigma$ by translation

FA1:

T inv, $\|S\|\leq\|T^{-1}\|^{-1}\Rightarrow T-S$ inv if $T,S\in L(X)$

S-W: linear sublattice w/ const func / subalgebra that seps points is dense in $C(K)$. lin subl = max, min $\in L$

subalg: cont. const funcs $f\cdot g$.

separable: $\exists D$ countable & dense
closed under equiv norms, isometric isomorphism

Hahn-Banach: X normed, Y subsp X, $f\in Y^*$ $\exists F\in X^*$ st $F|_Y=f$, $\|f\|=\|F\|$

$X^*:=L(X,\mathbb{F})$

if $1/p+1/q=1$, $L^p(\Omega)^*\cong L^q(\Omega)$ by $f\mapsto\int fg$
$\ell^p(\Omega)^*\cong\ell^q(\Omega)$ $x_n\mapsto\sum x_ny_n$

so L^p, ℓ^p reflexive for $p\in(1,\infty)$.

dual map: $T':Y^*\to X^*$ $T'(f)=x\mapsto f(T(x))$ $\|T'\|=\|T\|$

Heine-Borel: clsd & bdd in $\mathbb{R}^n\Leftrightarrow$ compact

ℓ_p counterexamples: e.g. truncated seqs, though may not conv to seq under sup norm.

DCT: $f_n\to f$ pwise, $|f_n(x)|\leq g(x)$, g integ $\Rightarrow f$ integ, $\|f_n-f\|\to0$

MCT: $f_n\uparrow f$ pwise, $\int f_n\uparrow\int f$, f integ

Fatou: $\int\liminf\leq\liminf\int f_n$
$\limsup\int f_n\leq\int\limsup f_n$ if $f_n\leq g$ integ.

Jensen $f(\int g(x)dx)\leq\int f(g(x))dx$ f convex.

Examples:

- $T(a_n)=(a_1,a_2/2,\ldots a_i/i)$ on ℓ^2 is injective, not surjective $[1,1/2,1/3,\ldots]$ and self adjoint so $Im\,T^*\neq\ell^2=(\ker T)^\perp$. $\sigma(T)=\sigma_{ap}(T)=0$, $1/k: k\in\mathbb{N}$; $\sigma_p(T)=1/k$, $\sigma_c=0$, $\sigma_r=\emptyset$ ↳ e_k $0\,0\,0\,1\,0\,0\,0$
- Right shift on $\ell^2(\mathbb{Z})$: unitary, $\sigma(R)=\sigma_{Ap}(R)=\sigma_c(R)=$ unit circle, $\sigma_p=\sigma_r=\emptyset$, same for left shift.
- on $L^2(\mathbb{R})$, $M_h:=f\mapsto f\times h\leftarrow$ pwise mult, $h\in L^\infty(\mathbb{R})$, real-valued
$\sigma(M_h)=\sigma_{Ap}=\lambda: h^{-1}(\lambda-\varepsilon,\lambda+\varepsilon)$ has pos. measure $\forall\varepsilon>0$ = ess. range
$\sigma_p=\lambda: h=\lambda$ has pos measure
$\sigma_r=\emptyset$, $\sigma_c=\sigma_{Ap}\setminus\sigma_p$.

Basic Pythagoras: $\|x\|+\|y\|=\|x+y\|$ if $x\perp y$. [in Hilb]
w/ projections, $Im(I-P)\subseteq\ker P$.