Tips: equiv norms $\Rightarrow$ both banach/not, but *not* for Hilbert/not; don't worry about integrability

X = vect spc over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. $(X, \|\cdot\|)$ a **normed space** if a) $\forall x \in X$ $\|x\| \geq 0$, and $\|x\| = 0 \iff x = 0$, b) $\|\lambda x\| = |\lambda|\|x\|$, c) $\|x+y\| \leq \|x\| + \|y\|$. every norm $\|\cdot\|$ induces a metric $d(x,y) = \|x-y\|$

Banach space: complete normed space - i.e. every Cauchy sequence in X converges.

for Banach $(X, \|\cdot\|_X)$ and a subspace $Y \subset X$, $(Y, \|\cdot\|_X)$ is complete = Banach $\iff$ Y is closed in X

Hilbert space: an *inner product space* $(X, <\cdot,\cdot>)$ complete wrt the norm $\|x\| = \sqrt{<x,x>}$

Examples: $p \in [1,\infty]$: $(\mathbb{R}^n, \|\cdot\|_p)$ or $(\mathbb{C}^n, \|\cdot\|_p)$, $\|x\|_p := (\sum_i |x_i|^p)^{1/p}$ for $1 \leq p < \infty$ / $\|x\|_\infty := \sup_i |x_i|$ $(\ell_p, \|\cdot\|_p)$, where $\ell_p := \{(x_j)_{j\in\mathbb{N}} : \sum_{j=1}^\infty |x_j|^p < \infty\}$ w/ $\|x\|_{\ell_p} := (\sum_{j=1}^\infty |x_j|^p)^{1/p}$ for $1 \leq p < \infty$

$\ell_\infty := \{(x_j)_{j\in\mathbb{N}}$ bdd$\}$ w/ $\|x\|_{\ell_\infty} := \sup_j |x_j|$ function spcs: $(L^p(\Omega), \|\cdot\|_{L^p})$ for $\Omega \subseteq \mathbb{R}$, intvl/meas. $\subseteq \mathbb{R}^n$

$\mathcal{L}^p := \{f : \Omega \to \mathbb{R}$ measurable st $\int_\Omega |f|^p dx < \infty\}$with $\|f\|_{L^p} := (\int_\Omega |f|^p dx)^{1/p}$ for $1 \leq p < \infty$

$\mathcal{L}^\infty := \{f : \Omega \to \mathbb{R}$ meas st $\exists M$ $|f| < M$ a.e.$\}$with $\|f\|_{L^\infty} := \operatorname{ess\,sup}|f| := \{\inf M : |f| \leq M$ a.e.$\}$

only actually norms on $L^p(\Omega) := \mathcal{L}^p/\sim$ - $f \sim g \iff f = g$ a.e.

Holder's ineq: $f \in L^p(\Omega), g \in L^q(\Omega)$ st $1/p + 1/q = 1$ then fg is int and $|\int_\Omega fg\,dx| \leq \|f\|_{L^p}\|g\|_{L^q}$ [1) log concave $\Rightarrow 1/p \log s + 1/q \log t \leq \log(s/p + t/q)$ 2) exponentiate, 3) $s = |f(x)|^p/\|f\|_p^p$ 4) integrate]

$\Omega \subseteq \mathbb{F}$ / $\mathbb{F}^n$, $\mathcal{F}^b(\Omega) := \{f : \Omega \to \mathbb{F}$ bdd$\}$, $C_b(\Omega) := \{f : \Omega \to \mathbb{F}$ bdd, cont$\}$ Banach [same prf, lim insd $\|\|$]

prod of normed spc w/ $\mathbb{R}^2$ norm, normed spc: Banach $\iff$ (abs conv $\|\cdot\| \Rightarrow$ conv) $[\|x_{n_j} - \| \leq 2^{-j}]$

Bounded Linear Operators

$T : X \to Y$ is a **BLO** if T is linear and $\exists M \in \mathbb{R}$ st $\forall x \in X$ $\|Tx\|_Y \leq M\|x\|_X$.

$L(X,Y) := \{T : X \to Y \mid T$ is a BLO$\}$ is a Banach space with the **operator norm**: [use sup for prfs!!]

$\|T\|_{L(X,Y)} := \inf\{M : \forall x \in X$ $\|Tx\|_Y \leq M\|x\|_X\} = \sup_{x\in X, x\neq 0} \|Tx\|/\|x\| = \sup_{x\in X, \|x\| = \text{ or } \leq 1} \|Tx\|$

linear $T : X \to Y$ 2x normed, TFAE: 1) Lipschitz cont, 2) continuous, 3) cont at 0 4) $T \in L(X,Y)$ [cont at 0 $\Rightarrow$ BLO: ※, rescale seq $\to 0$] [$\downarrow$: w/ S_n conv, $(ID - T)S_n = I - A^{n+1}$]

$T \in L(X), \|T\| < 1$, $(Id - T)^{-1} := \sum_{j=0}^\infty T^j \in L(X)$; [$\downarrow$: $T - S = T(I - T^{-1}S)$, terms inv, 2nd by above]

inv $T \in L(X), S \in L(X)$ st $\|S\| < \|T^{-1}\|^{-1}$, $T - S$ inv

Finite dimensional normed spaces

all norms on $\mathbb{R}^m$, fin-dim spaces equiv; [$\mathbb{R}$: comp. to 2-norm w/ C-S, rev: ※: seq x_n st $\|x_n\|_2 \geq n\|x_n\|$, unit sphere compact, fin-dim: basis $\to$ isomet isomorphs Q...]

all lin maps from fin-D normed spcs are BLOs $[\|x\|_T := \|x\|_X + \|Tx\|_Y]$; all m-D normed space are complete [Q, Q^{-1} $\uparrow, \mathbb{R}^n$ cmplt] [$\downarrow$ 1 $\to$ 2: $Q(Y)$ cpct, Q^{-1} cont, 3 $\to$ 1: ※lin indep x_k..., $Y_k :=$ span$(x_1 ... x_k)$, Riesz-lem w/ $Y_k \subseteq Y_{k+1}, 1/2$ gives seq $y_k \in Y_{k+1} \cap S$, isn't Cauchy, so S not seq compact]

TFAE: 1. $\dim(X) < \infty$, 2. $Y \subset X$ bdd, closed $\Rightarrow$ Y compact, 3. S =unit sphere is compact Riesz-l: closed $Y \subsetneq X$ normed, $\forall \epsilon > 0\, \exists x \in S$ unit sphr: dist$(x,Y) \geq 1 - \epsilon$ [$x' \in X/Y, d \leq \|x' - y'\| \leq \frac{d}{1-\epsilon}$, $x' - y'$]

Density & Stone-Weierstrass

$D \subset X$ is **dense** if $\overline{D} = X \iff$ approaches all points $\iff$ near all points [$\downarrow$ $y_n = x_n$ n odd, z_n even]

Y a dense subspace of $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ is Banach: $T \in L(Y,Z)$ has a unique extension $\tilde{T} \in L(X,Z)$.

$K \subseteq \mathbb{R}^n$ compact, $C(K) := C(K,\mathbb{R})$ w/ sup norm. $D \subseteq C(K)$, **separates points** if $\forall p,q \in K, p \neq q$: $\exists g \in D$ st $g(p) \neq g(q)$ ($\iff$ $= 0, = 1$ resp) **linear sublattice** if $(f,g \in D \Rightarrow \max, \min \in D)$ $\iff$ $(f \in D \Rightarrow |f| \in D)$ **S-W w/ lattices:** L lin subl., const funcs $\in L$, seps points $\Rightarrow L$ dense in $C(K)$

Prf lemma: $\forall f \in C(K)$, L sat S-W, $\forall p,q \in K$ $\exists f_{p,q} \in L$ st $f_{p,q}(p) = f(p)$ and $f_{p,q}(q) = f(q)$, and $\forall \epsilon > 0$ $\exists$open neighbourhood $U_{p,q}^\epsilon$ of $\{p,q\}$ in K st $|f - f_{p,q}| < \epsilon$ on $U_{p,q}^\epsilon$ [prv: sep points] [S-W prf: open cover of $U_{p,q}^\epsilon$'s for fixed p - finite subcover. $g_p := \min(f_{p,q_i})$. $g_p - f < \epsilon$. g_p's cont, ngbhrhood $V_p := (f - g_p)^{-1}(-\epsilon, \epsilon)$ of p (nb. $g_p(p) = f(0)$) st $g_p > f + \epsilon$. finite subcvr $p_1 ... p_k$ of V_p's, $g := \max_k$ g_{p_k}'s]

Polynomial approximation theorem: the space of polys is dense (w/ uniform conv) in $C(K)$.

subalgebra: $A \subseteq C(K)$ cont. const funcs, $f, g \in A \Rightarrow f \cdot g \in A$.

S-W w/ subalgebras: A subalgebra which separates points $\Rightarrow$ dense in $C(K)$ [$\overline{A}$ is a subalg & thus lin subl so $\overline{A}$ dense in $C(K) \Rightarrow \overline{A} = C(K)$.] closed subalgebra is linear sublattice [$f_{k+1} := f_k + 1/2(f^2 - f_k^2)$ incr conv to $|f|$ pwise and unif w/ $\to$] $K \subseteq M$ compact $\subseteq (M,d)$, $g_n : K \to \mathbb{R}$ decr seq of cont funcs, pwise $\to 0 \Rightarrow g_n \to 0$ unif. [$F_n = \{x : g_n(x) \geq \epsilon\}$, intsection empty, so 1 empty]

$\forall 1 \leq p < \infty$, any compact $K \subseteq \mathbb{R}^n$, $C^\infty(K)$ = smooth functions is dense in $L^p(K)$ [no proof]

Separability

$(X, \|\cdot\|)$ is **separable** if $\exists D \subseteq X$ which is countable and dense in X. [$\downarrow$ as $\mathbb{Q}^n$ dense, countbl in $\mathbb{R}^n$]

Separability is closed under norm equiv, isometric isomorphism, & all fin-dim normed spaces are separable.

$(\ell^\infty, \|\cdot\|_\infty)$ and $L^\infty(\Omega)$ for any non-empty $\Omega \subseteq \mathbb{R}^n$ are inseparable. [$\{0,1\}^\mathbb{N}$ uncountable, dist btwn =1]

$Y \subseteq X$ a subspace, D dense in $(Y, \|\cdot\|_X)$ & Y dense in (X) (w/ $\|\cdot\|_X$)$\Rightarrow$ D dense in X.

If $\exists S$ countable, span$(\overline{S})$ dense in $X \Rightarrow X$ separable. [rational lin combs of S dense: $\epsilon/3$ & countable]

$C(K)$ and $L^p(K)$ are separable for any compact set $K \subseteq \mathbb{R}^n$, $\ell^p(\mathbb{F})$ is separable, all for $1 \leq p < \infty$ [monomials countable, Weier: span dense. basis $e^{(k)}$ - span dense w/ cutoff seqs. char funcs of intvals w/ rational ends - step funcs dense] If $(X, \|\cdot\|_X)$ is separable and Y is a subspace of X, then $(Y, \|\cdot\|_X)$ is separable [$y_{k,n}$ st $\|x_k - y_{k,n}\| \leq \text{dist}(x_k, Y) + 1/n$, set of y's $\forall k, n$ is dense.]

dual space: $X^* := L(X,\mathbb{F})$ with *operator norm.* Always complete. **Hahn-Banach**
H-B for bdd extension: X *normed* space, $Y \leqslant X, f \in Y^*, \exists F \in X^*$, st $F|_Y = f, \|F\|_{X^*} = \|f\|_{Y^*}$.
Have $\|F\|_{X^*} \geq \|f\|_{Y^*}$ as $Y \subset X$, only need $F(x) \leq p(x) := \|f\|\|x\|$. **sublinear**: $p(x+y) \leq p(x) + p(y)$
& $p(\lambda x) = \lambda p(x)$ for $\lambda \geq 0$. **real H-B for sublinear** $f(y) \leq p(y)$ sublin, $F(x) \leq p(x)$, not BLO [ONLY
linear f, separable, lma: $X = \text{span}(Y \cup \{x_0\})$: $x = y + \lambda x_0$ uniq, def $\tilde{f}_r(y + \lambda x_0) = f(y) + \lambda r$, bdds on r
sublin $\leq \inf_v(p(v + x_0) - f(v))$, $-\lambda$. complex: extend real part to F_1, $F(x) := F_1(x) - iF_1(ix)$]
X *normed*: $\forall x \in X\backslash\{0\}$, , $\exists f \in X^*$ w/ $\|f\| = 1$, st $f(x) = \|x\|$. [$Y = \text{span}(x), g(\lambda x) = \lambda\|x\|$]
$\forall x \in X : \|x\|_X = \sup_{f \in X^*, \|f\|_{X^*}=1} |f(x)|$ [use ↑] [for ↓, use ↑ w/ $x' = x - y$]
For any $x \neq y$ in a normed space, ∃a linear functional $f \in X^*$ that separates them, i.e. $f(x) \neq f(y)$.
$f : X \to \mathbb{F}, f \neq 0$ linear: $\forall x_0 \in X$ st $f(x_0) \neq 0$, $\text{span}(\ker(f) + \{x_0\}) = X$ [$\lambda : f(x)/f(x_0)$,$x - \lambda x_0 \in \ker(f)$]
$Y \leqslant X$ Banach (Y proper, closed) $\forall x \in X\backslash Y\ \exists f \in X^*$ st $\|f\| = 1$, $f|_Y = 0$, $f(x) = \text{dist}(x, Y) > 0$.
annihilator of $A \subseteq X$ is $A^\circ := \{f \in X^* : f|_A = 0\}$ [↑: $g(y + \lambda x_0) := \lambda d(x_0, Y))$ on $\text{span}(Y \cup \{x_0\})$]
annihilator (in dual) of $T \subseteq X^*$ is $T_\circ := \{x \in X : \forall f \in T, f(x) = 0\} = \bigcap_{f \in T} \ker(f)$ [bdd as $y/|\lambda| \in Y$]
In normed X, $S \subseteq X$, $T \subseteq X^*$: • $\overline{S} = (S^\circ)_\circ$ [$\subseteq$ RHS clsd, defs $\supset$ ※, $\exists f\ f(x) \neq 0, = 0$ on $S \Rightarrow$ ※$x \notin$RHS]
• $\text{span}(S)$ dense $\iff$ $S^\circ = \{0\} \subseteq X^*$; [conv: sup. not] • $\text{span}(T)$ dense in $X^* \Rightarrow T_\circ = \{0\} \subseteq X$. [※]

Dual spaces, second duals & completion
for $f : X \to \mathbb{F}$, linear, on normed X, $\ker(f)$ is closed $\iff$ $f \in X^*$. [$\delta^{-1}, \delta = d(x_0, \ker), x_0 \notin \ker$]
Riesz R: X Hilb, $\iota : X \to X^*$ defined by $\iota(x)(y) = \langle x, y\rangle$ is an isometric isomorphism. [no prf]
Examples: find isometric isomorphism to known space. for $1 \leq p < \infty$: $q \in (1, \infty]$ st $1/p + 1/q = 1$
L^p: $(L^p(\Omega))^* \cong L^q(\Omega)$, $\iota : L^q(\Omega) \to (L^p(\Omega))^*$ is $\iota(f) = g \mapsto \int_\Omega f \cdot g \, dx \in \mathbb{R}$. [Holder, no surj, $\|\iota f\| \geq \|f\|$]
$\ell^p(\mathbb{R})$: $(\ell^p(\mathbb{R}))^* \cong \ell^q(\mathbb{R})$, $\iota : \ell^q(\mathbb{R}) \to (\ell^p(\mathbb{R}))^*$ where $\iota(x) = y \mapsto \sum_{j=1}^\infty x_j y_j \in \mathbb{R}$.[$g := |f|^{q-2} f$, sep $p = 1$]
second dual X^{**} of normed X exists as dual normed. $i : X \to X^{**}$; $i(x)(f) := f(x)$ isometric lin map.
X is **reflexive** if $i(X) = X^{**}$ (normally a proper subsp) - e.g. ℓ^p, L^p for $1 < p < \infty$
X isometrically isomorphic to $i(X)$, a dense subsp of (Banach) $(\overline{i(X)}, \|\cdot\|_{X^{**}})$. **completion** of X:
complete space X^* + isometry ϕ st $\phi(X)$ dense in X^*
Dual operators: $T : X \to Y$ lin, X, Y vect spc over $\mathbb{F}$, $X' := \{f : X \to \mathbb{F} \text{ lin}\}$, **dual map**: $T' : Y^* \to X^*$
is $T'(f) = x \mapsto f(T(x))$ X, Y normed, $T \in L(X, Y) \Rightarrow T' \in L(Y^*, X^*)$, $\|T'\|_{L(Y^*, X^*)} = \|T\|_{L(X,Y)}$
$T \in L(X)$ is invertible if bij (=alg inv exists) & $T^{-1} \in L(X)$ **Spectral Theory**
given T^{-1} exists, $T^{-1} \in L(X)$ $\iff$ $\exists \delta > 0$ st $\forall x \in X\ \|Tx\| \geq \delta\|x\|$. [prf ↓: TX clsd: Cauchy + Banach]
Banach X,$T \in L(X)$: $\exists \delta > 0$ st $Tx\| \geq \delta\|x\| \Rightarrow$ 1) T inj 2) $TX \subseteq X$ clsd 3) TX dense in $X \Rightarrow T$ inv.
Normed X ,$S, T \in L(X), ST = TS$ inv, S and T both inv. [※: T not surj/no $\delta \Rightarrow$ no δ for ST]
$(X, \|\cdot\|)$ be a normed spc over $\mathbb{C}$, $T \in L(X)$: **resolvent set**: $\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda\text{Id invertible}\}$,
spectrum: $\sigma(T) := \mathbb{C}\backslash\rho(T)$ **resolvent operator** for $\lambda \in \rho(T)$: $R_\lambda(T) := (T - \lambda\text{Id})^{-1} \in L(X)$
$\lambda \in \sigma(T)$ if ≥ 1 of: 1) $T - \lambda\text{Id}$ not inj 2) $\neg\exists\delta > 0$ st $\forall x \in X\ \|Tx - \lambda x\| \geq \delta\|x\|$ 3) $T - \lambda\text{Id}$ not surj
point spectrum = set of **eigenvalue**s of T: $\lambda \in \mathbb{C}$ if $\exists x \in X, x \neq 0$ st $Tx = \lambda x$.
approx point spc =**approx e-val**s of T: $\lambda \in \mathbb{C}$ if $\exists(x_n)_{n\geq1} \in X$ with $\|x_n\| = 1$ & $\|Tx - \lambda x\| \to 0$. 2) ↑
$\sigma_P(T) \subseteq \sigma_{AP}(T) \subseteq \sigma(T)$. note: $T \in L(X)$ for X fin-dim has $\sigma(T) = \sigma_P(T)$ [Rank-N, cont]
if X Banach: $\rho(T)$ is open, $\rho(T) \ni \lambda \mapsto R_\lambda(T)$ is analytic=repr. as power series around any point
• $\sigma(T)$ non-empty, compact [clsd+bd], closed & $\forall\lambda \in \sigma(T) : |\lambda| \leq \|T\|_{L(X)}$ [>: $R_\lambda = -\lambda(I - \lambda^{-1}T)$ inv]
• $\lambda \in \sigma(T), j \in \mathbb{N} \Rightarrow \lambda^j \in \sigma(T^j) \Rightarrow |\lambda|^j \leq \|T^j\|$ [$\lambda \notin \sigma(T^j) \Rightarrow R_\lambda(T^j) = R_\lambda(T)(T^{j-1} + \lambda T^{j-2}...)$ inv]
• **spectral radius**: $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\} = \lim_{j\to\infty} \|T^j\|^{1/j} = \inf_{j\in\mathbb{N}} \|T^j\|^{1/j}$ [no proof]
• for any complex poly p, $\sigma(p(T)) = p(\sigma(T)) := \{p(\lambda) : \lambda \in \sigma(T)\}$ [factorise $p(T) - \mu I$, $ST = TS$ lem]
• $\sigma(T) = \sigma_{AP}(T) \cup \sigma_P(T')$ - NB dual [$\sigma_P(T') \subseteq \sigma(T)$: take e-vect f, $f(Tx - \lambda x) = 0 \Rightarrow R_\lambda$ not surj.
$\sigma(T) \setminus \sigma_{AP}(T) \cup \sigma_P(T')$: im clsd & proper $\Rightarrow \exists f$ BLO $f|_Y = 0$, is e-vect for T, λ] **Other**
Heine-Borel clsd & bdd in $\mathbb{R}^n$ $\iff$ compact | **isometric isomorphism**: linear T if it is isometric, so
$\|Tx\| = \|x\|$, & an isomorphism=biject (so has isomet. inverse) | **support** of func: $\overline{\{x : g(x) \neq 0\}}$
example seqs: try truncated seqs 1st, not $j \mapsto j^{-1+1/n}$ - can trunc at $2n$ or $2n - 1$ - whatever fits seq
care w/ truncs, sups: trunc seq may not conv under sup norm to orig
Spectra: $\sigma_p \neq \sigma_{ap}$ example: $T(x) = (x_j/j)_{j\in\mathbb{N}}$ on ℓ_∞: $\lambda = 0$ | left shift on $\text{span}\{e^{(j)}\}$: $\sigma_P = \emptyset$ as
$a_j = \lambda^j \notin S$, but $\sigma_{AP} = |\lambda| < 1$ as can take cutoffs. | use spectra of poly rule to simplify
counterexamples: $X \subseteq Y \subseteq Z$ - 2 could be same set w/ diff norm. | incomplete normed space: e.g.
$\{e^{(j)}\} \subseteq \ell_p$, or finite span of | $n\mathbf{1}[1/n, n]$ | $\int 1/x^p$ on $[0, 1]$ iff $p < 1$, on $[1, \infty]$ iff $p > 1$, sum $1/j^p$ $p > 1$
NB convergence doesn't just mean conv. to 0 | if have unbdd seq: can take subseq $a_{n_j} \geq j$ (by ind) |
$\sum 1/(n(\log n)^\beta)$ converges $\iff$ $\beta > 1$ | if norm def as an integral in L^1 (/sum in ℓ_1), use L^1 complete,
def limit from conv in L^1. | consider taking pointwise limits inside $|\cdot|$ | isomet isomorph duals: use
separability, standard results, c_{00} = finitely many non-zero, dual is space of all seqs, dual of c_0 is ℓ_1,
”respacing of sequences” - e.g. only odd terms still equiv under dual to normal