Functional Analysis I

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1 Banach spaces

Let X be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

$\| \cdot \| : X \to \mathbb{R}$ is a function if:

- $\forall x \in X \ \|x\| \geq 0$, and $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

a pair $(X, \| \cdot \|)$ is a **normed space**.

every norm $\| \cdot \|$ induces a metric $d(x, y) = \|x - y\|$, so everything from Part A Metric spaces also applies.

Notes on Part A content:

- just translate d to $\| \cdot \|$, but ensure that you don't confuse norms.
- Cauchy sequence: $\forall \varepsilon > 0 \ \exists N$ st $\forall n, m \geq N \ \|x_n - x_m\| < \varepsilon$
- equivalent norms...
  - so equiv => equal banach, but two equiv norms are not necessarily both hilbert spaces
- subspace is a normed space by restricting the norm

A **Banach space** is a complete normed vector space - i.e. every Cauchy sequence in X converges.

Given $(X, \| \cdot \|_X)$ is a Banach space, and $Y \subset X$ is a subspace, $(Y, \| \cdot \|_X)$ is a Banach space $\iff$ Y is closed in X

An **inner product space** $(X, < \cdot, \cdot >)$ is called a **Hilbert space** if it is complete wrt to the norm $\|x\| = \sqrt{< x, x >}$ (more detail in FA2)

Thus, Hilbert space $\implies$ Banach space $\implies$ complete metric space.

1.1 Examples of metric spaces

Given $p \in [1, \infty]$:

$(\mathbb{R}^n, \| \cdot \|_p)$ or $(\mathbb{C}^n, \| \cdot \|_p)$, where

$$\|x\|_p := \left(\sum_i |x_i|^p \right)^{1/p} \quad \text{for } 1 \leq p < \infty$$

$$\|x\|_\infty := \sup_i |x_i|$$

sequence spaces: $(\ell_p, \| \cdot \|_p)$, where

$$\ell_p := \left\{ (x_j)_{j \in \mathbb{N}} : \sum_{j=1}^{\infty} |x_j|^p < \infty \right\} \text{ for } 1 \leq p < \infty$$

$$\ell_\infty := \left\{ (x_j)_{j \in \mathbb{N}} : (x_j) \text{ is a bounded sequence} \right\}$$

$$\|x\|_{\ell_p} := \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} \text{ for } 1 \le p < \infty$$

$$\|x\|_{\ell_\infty} := \sup_j |x_j|$$

function spaces: $(L^p(\Omega), \|\cdot\|_{L^p})$

given $\Omega \subseteq \mathbb{R}$ is an interval/a measurable subset of $\mathbb{R}^n$, consider first:

$$\mathcal{L}^p := \left\{f : \Omega \to \mathbb{R} \text{ measurable st } \int_\Omega |f|^p dx < \infty\right\} \text{ for } 1 \le p < \infty$$

$$\mathcal{L}^\infty := \{f : \Omega \to \mathbb{R} \text{ measurable st } \exists M \ |f| < M \text{ a.e.}\}$$

(note that we only consider measurable functions and the Lebesgue integral/measure so no need to worry about integrability)

Their norms are:

$$\|f\|_{L^p} := \left(\int_\Omega |f|^p dx\right)^{1/p} \text{ for } 1 \le p < \infty$$

$$\|f\|_{L^\infty} := \operatorname{ess\,sup} |f| := \{\inf M : |f| \le M \text{ a.e.}\}$$

These two functions are only actually norms on $L^p(\Omega) := \mathcal{L}^p/\sim$ equipped with $\|\cdot\|_{L^p}$, where $f \sim g \iff f = g$ a.e.

bounded functions

cont bounded functions

cont functions on compact sets

products

sum of subspaces

quotient spaces

completeness

of various spaces

results for completeness

NORMS

Holder's inequality:....

2 Bounded linear operators

Given $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$, $T : X \to Y$; T is a **bounded linear operator** if T is linear and $\exists M \in \mathbb{R}$ st $\forall x \in X \ \|Tx\|_Y \le M\|x\|_X$.

$L(X,Y) := \{T : X \to Y \mid T \text{ is a bounded linear operator}\}$, with the **operator norm** $\|T\|_{L(X,Y)} := \inf\{M : \forall x \in X \ \|Tx\|_Y \le M\|x\|_X\}$ is a normed space.

$$\|T\|_{L(X,Y)} = \sup_{x\in X, x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x\in X, \|x\|=1} \|Tx\| = \sup_{x\in X, \|x\|\le 1} \|Tx\|$$

and $\|Tx\| \le \|T\|\|x\|$. Note T is not actually a bounded function.

Given T is a linear function between normed spaces, TFAE: T is Lipschitz cont, T is cont, T is cont at 0 and $T \in L(X,Y)$

$L(X,Y)$ is a Banach space

composition of blos is a BLO

3 Finite dimensiona normed spaces

all norms on $\mathbb{R}^m$ are equivalent, and all norms on finite dimenionsal spaces are equivalent

if $T : X \to Y$ is a linear map between normed spaces, and X is fin-dim, then T is a BLO.

An m-dimensional normed space $(X, \|\cdot\|)$ is homeomorphic to $\mathbb{F}^m$.

every finite dimensional normed space is complete, because the $\mathbb{F}^m$ are $\forall m$, and so are finite dimensional subspaces of normed spaces.

TFAE:

1. $\dim(X) < \infty$
2. $Y \subset X$ bounded and closed $\implies Y$ compact
3. $S := \{x \in X : \|x\| = 1\}$ is compact

4 Density & Stone-Weierstrass

$D \subset X$ is **dense** if

$$\begin{aligned} \overline{D} = X &\iff \forall x \in X \; \exists (y_n) \in D^{\mathbb{N}} \text{ st } y_n \to x \text{ as } n \to \infty \\ &\iff \forall x \in X \; \forall \varepsilon > 0 \; \exists y \in D \; \|x - y\| < \varepsilon \end{aligned}$$

If Y is a dense subspace of $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ is a Banach space, then $T \in L(Y, Z)$ has a unique extension $\tilde{T} \in L(X, Z)$.

We now consider a compact subset $K \subseteq \mathbb{R}^n$, and $C(K) = C(K, \mathbb{R})$, the space of continuous real-valued functions, with the sup norm.

$D \subseteq C(K)$ **separates points** if $\forall p, q \in K, p \neq q$: $\exists g \in D$ st $g(p) \neq g(q)$ or $\exists g \in D$ st $g(p) = 0$ and $g(q) = 1$

$D \subseteq C(K)$ is a **linear sublattice** if $f, g \in D \implies \max(f, g), \min(f, g) \in D$ or equivalently if $f \in D \implies |f| \in D$

Stone-Weierstrass w/ lattices: if L is a linear sublattice, contains the constant functions, and separates points in K, then L is dense in $C(K)$

Lemma for proof: for any $f \in C(K)$, $L \subseteq C(K)$ containing the constant functions, separates points, $\forall p, q \in K \; \exists f_{p,q} \in L$ st $f_{p,q}(p) = f(p)$ and $f_{p,q}(q) = f(q)$, and $\forall \varepsilon > 0$ $\exists$an open neighbourhood $U^\varepsilon_{p,q}$ of $\{p, q\}$ in K st $|f - f_{p,q}| < \varepsilon$ on $U^\varepsilon_{p,q}$

Polynomial approximation theorem: the space of polynomials is dense (i.e. under uniform convergence) in $C(K)$.

$A \subseteq C(K)$ is a **subalgebra** if A contains the constant functions and $f, g \in A \implies f \cdot g \in A$

A closed subalgebra of $C(K)$ is a linear sublattice [4.7]

Stone-Weierstrass for subalgebras: If A is a sublagebra which separates points, A is dense in $C(K)$

Generalisatation of Dini's theorem: Given $K \subseteq M$ is a compact subset of a metric space (M, d) and $g_n : K \to \mathbb{R}$ is a decreasing sequence of continuous functions, converging pointwise to 0. Then $g_n \to 0$ uniformly. [4.8]

for any $1 \leq p < \infty$, and any compact $K \subseteq \mathbb{R}^n$, the space $C^\infty(K)$ of smooth functions is dense in $L^p(K)$ [4.9, no proof]

5 Separability

A normed space $(X, \|\cdot\|)$ is **separable** if $\exists D \subseteq X$ which is countable and dense in X.

Separability is closed under equivalence of norms and isometric isomorphism.

Every finite dimensional normed space is separable.

E.g.:$(\ell^\infty, \|\cdot\|_\infty)$ and $L^\infty(\Omega)$ for any non-empty $\Omega \subseteq \mathbb{R}^n$ are inseparable.

Given: $(X, \|\cdot\|_X)$ is a normed space, $Y \subseteq X$ a subspace (with norm $\|\cdot\|_X$), if D is dense in $(Y, \|\cdot\|_X)$ and Y is dense in $(X, \|\cdot\|_X)$, then D is dense in X.

IF there is a countable set S st $\text{span}(\overline{S})$ is dense in a normed space X, then X is separable.

$C(K)$ and $L^p(K)$ are separable for any compact set $K \subseteq \mathbb{R}^n$, $\ell^p(\mathbb{F})$ is separable, all for $1 \leq p < \infty$

If $(X, \|\cdot\|_X)$ is separable and Y is a subspace of X, then $(Y, \|\cdot\|_X)$ is separable

6 Hahn-Banach

The **dual space** of a normed space $(X, \|\cdot\|)$ is $X^* := L(X, \mathbb{F})$ with the operator norm $\|f\|_{X^*} := \sup_{x \in X} \frac{|f(x)|}{\|x\|}$. Note that it is always complete, and $\forall f \in X^*, x \in X$ $|f(x)| \leq \|f\|\|x\|$

Theorem 1. *Hahn-Banach for bounded extension*

Given a real/complex normed space X, a subspace $Y \subset X$, $f \in Y^*$, there is an extension $F \in X^*$ of f - i.e. $F|_Y = f, \|F\|_{X^*} = \|f\|_{Y^*}$.

Note that we already have $\|F\|_{X^*} \geq \|f\|_{Y^*}$, as $Y \subset X$, so all we need to prove is $F(x) \leq p(x) := \|f\|\|x\|$.

We generalise this by considering all p that are **sublinear** - i.e. $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for $\lambda \geq 0$. Thus:

Theorem 2. *real Hahn-Banach for sublinear functions*

Given a normed space X, a subspace $Y \subset X$ and $p : X \to \mathbb{R}$ sublinear, with $f \in Y^*$ st $f(y) \leq p(y)$ $\forall y \in Y$, then there is an extension $F \in X^*$ of f st $\forall x \in X$ $F(x) \leq p(x)$

Proof. if X is inseparable, not on the course. If it is, see notes, using the intermediate lemma for the case $X = \operatorname{span}(Y \cup \{x_0\}$ □

Applications

$\forall x \in X \backslash \{0\}$, where $(X, \|\cdot\|)$ is a normed space, $\exists f \in X^*$with $\|f\| = 1$, st $f(x) = \|x\|$.

On a normed space $(X, \|\cdot\|)$:

- $\forall x \in X :$ $\|x\|_X = \sup_{f \in X^*, \|f\|_{X^*}=1} |f(x)|$
- $\forall f \in X^* :$ $\|f\|_{X^*} = \sup_{x \in X, \|x\|_X=1} |f(x)|$

For any $x \neq y$ in a normed space there is a linear functional $f \in X^*$ that separates them, so $f(x) \neq f(y)$.

If $f : X \to \mathbb{F}, f \neq 0$ is linear for a normed space X, then $\forall x_0 \in X$ with $f(x_0) \neq 0$, $\operatorname{span}(\ker(f) + \{x_0\}) = X$

we can separate points from closed subspaces: for a proper closed subspace $Y \subsetneq X$ a Banach space, then $\forall x \in X \backslash Y$ $\exists f \in X^*$ with $\|f\| = 1$ st $f|_Y = 0$ and $f(x) = \operatorname{dist}(x, Y) > 0$.

the **annihilator** of $A \subseteq X$ is $A^\circ := \{f \in X^* : f|_A = 0\}$, and for $T \subseteq X^*$, $T_\circ := \{x \in X : \forall f \in T, f(x) = 0\} = \bigcap_{f \in T} \ker(f)$

In a normed space $(X, \|\cdot\|)$:

- for $S \subseteq X$, $\operatorname{span}(S)$ is dense $\iff$ $S^\circ = \{0\} \subseteq X^*$
- for $T \subseteq X^*$, $\operatorname{span}(T)$ is dense in X^* $\implies$ $T_\circ = \{0\} \subseteq X$.

for any $A \subseteq X$ normed, $\overline{A} = (A^\circ)_\circ$

7 Dual spaces, second duals & completion

for $f : X \to \mathbb{F}$, linear, on a normed space X, $\ker(f)$ is closed $\iff$ $f \in X^*$.

Riesz Representation theorem: for a Hilbert space X, the map $\iota : X \to X^*$ defined by $\iota(x)(y) = \langle x, y \rangle$ is an isometric isomorphism. [no proof]

Dual spaces of specific examples: we characterise them by isometric isomorphism to a known space.

- L^p for $1 \leq p < \infty$: let $q \in (1, \infty]$ st $1/p + 1/q = 1$. Then $(L^p(\Omega))^* \cong L^q(\Omega)$, and the bijective linear map is $L : L^q(\Omega) \to (L^p(\Omega))^*$ where $L(f) = g \mapsto \int_\Omega f \cdot g \, \mathrm{d}x \in \mathbb{R}$.
- $\ell^p(\mathbb{R})$ for $1 \leq p < \infty$: let $q \in (1, \infty]$ st $1/p + 1/q = 1$. Then $(\ell^p(\mathbb{R}))^* \cong \ell^q(\mathbb{R})$, and the bijective linear map is $L : \ell^q(\mathbb{R}) \to (\ell^p(\mathbb{R}))^*$ where $L(x) = y \mapsto \sum_{j=1}^{\infty} x_j y_j \in \mathbb{R}$.

The **second dual** of a normed space X is X^{**}, which exists as the dual space is itself a normed space.

$i : X \to X^{**}$ defined by $i(x)(f) := f(x)$ is an isometric linear map.

X is **reflexive** if $i(X) = X^{**}$ (normally it is a proper subspace) - e.g. ℓ^p, L^p for $1 < p < \infty$

X is isometrically isomorphic to $i(X)$, which is a dense subspace of the Banach space $(\overline{i(X)}, \|\cdot\|_{X^{**}})$, so we can consider any non-complete normed space as a dense subspace of a Banach space - this is the **completion** of X.

Dual operators

for any linear map $T : X \to Y$ for vector spaces X, Y over the same field, and $X' := \{f : X \to \mathbb{F} \text{ linear}\}$, then $T' : Y' \to X'$ is the **dual map** of T, defined by $T'(f) = x \mapsto f(T(x))$

Given X, Y are normed spaces, and $T \in L(X, Y)$, the dual map is well-defined and $T' \in L(Y^*, X^*)$, and $\|T'\|_{L(Y^*,X^*)} = \|T\|_{L(X,Y)}$

8 Spectral theory

$T \in L(X)$ is invertible if it is bijective (so has an algebraic inverse, which might not be continuous), and $T^{-1} \in L(X)$

If T is algebraically invertible, then it is invertible/$T^{-1} \in L(X) \iff \exists \delta > 0$ st $\forall x \in X\ \|Tx\| \geq \delta \|x\|$.

On a Banach space X, for $T \in L(X)$ st $\exists \delta > 0$ st $\forall x \in X\ \|Tx\| \geq \delta \|x\|$:

- T is injective
- $TX \subseteq X$ is closed
- if TX is dense in X, then T is invertible.

On a normed space X, for $S, T \in L(X)$ st $ST = TS$, ST is invertible (so $(ST)^{-1} \in L(X)$, then S and T are each invertible.

Spectrum & resolvent set

Let $(X, \|\cdot\|)$ be a normed space over the field $\mathbb{C}$.

For $T \in L(X)$:

- the **resolvent set** is $\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda \mathrm{Id} \text{ is invertible}\}$
  - $R_\lambda(T) := (T - \lambda \mathrm{Id})^{-1} \in L(X)$ is called the **resolvent operator** for $\lambda \in \rho(T)$
- the **spectrum** of T is $\sigma(T) := \mathbb{C} \backslash \rho(T)$

$\lambda \in \sigma(T)$ if at least one of:

1. $T - \lambda \mathrm{Id}$ is not injective
2. $\neg \exists \delta > 0$ st $\forall x \in X\ \|Tx - \lambda x\| \geq \delta \|x\|$
3. $T - \lambda \mathrm{Id}$ is not surjective

$\lambda \in \mathbb{C}$ is an **eigenvalue** of T if $\exists x \in X, x \neq 0$ st $Tx = \lambda x$. $\sigma_P(T) := \{\lambda : \lambda \text{ is an eigenvalue of } T\}$ is the **point spectrum**

$\lambda \in \mathbb{C}$ is an **approximate eigenvalue** of T if there exists a sequence $x_n \in X$ with $\|x_n\| = 1$ and $\|Tx - \lambda x\| \to 0$. This is identical to condition 2 above. $\sigma_{AP}(T) := \{\lambda : \lambda \text{ is an approximate eigenvalue of } T\}$ is the **approximate point spectrum**

$\sigma_P(T) \subseteq \sigma_{AP}(T) \subseteq \sigma(T)$.

Quick examples:

- any linear map $L : X \to X$ on a finite dimensional X has $\sigma(T) = \sigma_P(T)$, BECAUSE!!!!!
- $T(x) = (x_j/j)_{j \geq 1}$ where $T \in L(\ell^\infty)$ has $0 \in \sigma_{AP}(T)$ but not $\sigma_P(T)$.
- $T(x) = t \mapsto \int_0^t x(s)\,\mathrm{d}s$ in $L(C[0,1])$ has no eigenvalues, but $\sigma(T) = \{0\}$

For any $T \in L(X)$, X a complex Banach space:

- $\rho(T)$ is open, and the map $\rho(T) \ni \lambda \mapsto R_\lambda(T)$ is analytic - $\forall \lambda_0 \in \rho(T)$ there is a neighbourhood U of λ_0 in $\rho(T)$ and coefficients $A_j(\lambda_0, T) \in L(X)$ st $\forall \lambda \in U$
$$R_\lambda(T) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j A_j(\lambda_0, T)$$
- $\sigma(T)$ is non-empty, compact, closed and $\forall \lambda \in \sigma(T)\ |\lambda| \leq \|T\|_{L(X)}$
- for any $\lambda \in \sigma(T), j \in \mathbb{N}$, $\lambda^j \in \sigma(T^j)$, so $|\lambda|^j \leq \|T^j\|$
- $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$ is the **spectral radius**
- $r(T) = \lim_{j \to \infty} \|T^j\|^{1/j} = \inf_{j \in \mathbb{N}} \|T^j\|^{1/j}$
- for any complex poly p, $\sigma(p(T)) = p(\sigma(T)) := \{p(\lambda) : \lambda \in \sigma(T)\}$
- $\sigma(T) = \sigma_{AP}(T) \cup \sigma_P(T')$ - note dual here.