Candidate Number:

counting process: X_t $t \geq 0$ $X_t \in \mathbb{N}$, $s \leq t \Leftrightarrow X_s \leq X_t$ | holding times | equivalence pfs: (1⇒2) $\mathbb{P}(X_t=n)=\mathbb{P}(T_n \leq t)-\mathbb{P}(T_{n+1} \leq t)$, int by P first term, then cancels

Poisson process: #1: $X_t = \#\{n \geq 1 : T_n \leq t\}$ $T_n := \sum_1^n Z_k$, Z_k iid $\mathrm{Exp}(\lambda)$. indep incr: induct w/ cond prob, Markov prop; stat incr via PP(λ)

#2 transition prob: $X_t \sim \mathrm{Po}(\lambda t)$ $\forall t$, indep incr: $\{X_{t_k} - X_{t_{k-1}}, 1 \leq k \leq n\}$ indep for $0 = t_0 \leq t_1 \dots \leq t_n < \infty$ 2⇒1: $\mathbb{P}(X_{t_1}=k_1 \dots X_{t_n}=k_n)$ same as def 1 so ✓

#3 infinitesimals — stationary increments: $X_{t+s} - X_t \overset{d}{=} X_s$. 1,2 ⇒ 3: indep incr ✓, $o(h)$: stat incr, L'H. 3⇒2: diff eqs → same dist, stat incr = 2.

3a: indep increments, 3b) as $h \downarrow 0$, unif in t, $\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h)$, $\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$. pf: $\hat{Z}_1 := T_{k+1} - t = Z_{k+1} - (t - T_k)$, $\hat{Z}_n = Z_{k+n}$

Markov property: $X \sim PP(\lambda)$ given $X_t = k$, $(X_r)_{r \leq t}$, $(X_{t+s})_{s \geq 0}$ indep $(X_{t+s})_{s \geq 0} \sim PP(\lambda)$ starting at k. extend. mem. $L = t - T_k$ for $\tilde{Z}_1$

↳ equiv def: $\mathbb{P}(X_{t_n} = k_n \mid X_{t_1} = k_1 \dots X_{t_{n-1}} = k_{n-1}) = \mathbb{P}(X_{t_n} = k_n \mid X_{t_{n-1}} = k_{n-1}) \Leftrightarrow$ same, w/ t, K_n sets X_r, $\tilde{X}_s$ indep. constructed

Memoryless: $E \sim \mathrm{Exp}(\lambda)$ $\forall x, y \geq 0$ $\mathbb{P}(E > x + y \mid E > y) = \mathbb{P}(E > x)$ extend to $y = L$, an RV ≥ 0, indep of E. pf: $\mathbb{P}(E - L > u) = \int_a^\infty \mathbb{P}(Z - L > u \mid Z = z) \lambda e^{-\lambda z} dz \dots$

$X \sim PP$, cond on $\{X_t = n\}$, $T_1, \dots T_n \overset{d}{\sim}$ ordered sample of $\mathcal{U}[0,t]$. pf: compare densities, change variable to $Z_1 \dots Z_n$ NB: Exp ⇔ memoryless $= e^{-\lambda u}$ $\mathbb{P}(E > L)$ ✓

superposition $X + Y \sim PP(\lambda + \mu)$ thinning mark point in $X \sim PP(\lambda)$ w/ prob p – result is $PP(p\lambda)$ pf: generating fns

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Simple birth: $X_t = k + \#\{n \geq 1 : \sum_1^n Z_i \leq t\}$ $Z_i \sim \mathrm{Exp}(\lambda_{i+(k-1)})$ | $\mathbb{P}$ $M \sim$ by indep, $\mathbb{P}(E_k < \inf_{j \neq k} E_j) = \frac{\lambda_k}{\lambda_k + \sum_{j \neq k} \lambda_j}$ by indep. $K := \begin{cases} k & E_k < \inf_{j \neq k} \\ \perp \end{cases}$

competing exponentials: indep $E_i \sim \mathrm{Exp}(\lambda_i)$, $i \in I$ finite/countable. $M := \inf_I E_i \sim \mathrm{Exp}(\sum_I \lambda_i)$, $\exists$ rand K st $M = E_K$ w/ prob 1. $\mathbb{P}(K = k) = \lambda_k / \sum \lambda_i$ $\mathbb{P}(K = \perp) = 1 - \dots = 0$

cond on $K = k$, M indep of $\{E_j - M : j \neq k\} \sim \mathrm{Exp}(\lambda_j)$: finite $J \subseteq I \setminus \{k\}$ $\mathbb{P}(M \geq y, E_j - M > x_j : j \in J \mid K = k) = \mathbb{P}(\dots \mid E_k < \inf) = \mathbb{P}(E_k \geq y, E_j > E_k + x_j, E_i > E_k : i \in I \setminus (J \cup k)) / \mathbb{P}(E_k < \inf)$. cond on E_k, int

Yule process: $\lambda_n = n\lambda$, each individual gives birth after $\mathrm{Exp}(\lambda)$. $m(t) := \mathbb{E}[Y_t] = e^{\lambda t}$ for $t \geq 0$. $T = Z_1$: $m(t) = \mathbb{E}[Y_t 1_{T \leq t}] + \mathbb{E}[Y_t 1_{T > t}] \leftarrow = \mathbb{P}(T > t)$

↳ $X_t \sim \mathrm{Geom}(e^{-\lambda t})$ as $T_n := \sum^n Z_i \overset{d}{=} \max_{\leq n} E_i \sim$ iid $\mathrm{Exp}(\lambda)$. then diff, $m(0) = 1$. → ↳ $= \int_0^t \mathbb{E}[\cancel{X_t} \mid U = u] = (2m(t-u))$ $\times f_T(u) du$

Markov prop for SBPs: same as PPs, both defs | X_t SBP, λ_n, k: $T_n := \inf\{t \geq 0 : X_t = k + n\}$, $T_\infty := \lim_{n \to \infty} T_n = \sum^\infty Z_i$ pf: $\mathbb{E}[e^{-T_\infty}] = 0 \Rightarrow \mathbb{P}(T_\infty = \infty) = 1$

explosion possible if $\mathbb{P}(T_\infty < \infty) > 0$. $\mathbb{P}(\text{explosion}) = 1 \Leftarrow \sum_{i=k}^\infty 1/\lambda_i < \infty$ pf: $\mathbb{E}[T_\infty] = \sum^\infty \mathbb{E}[Z_i] = \sum 1/\lambda_i$ by Tonelli $\mathbb{P}(\text{explosion}) = 0 \Leftarrow \sum 1/\lambda_i = \infty$ $-\log \uparrow \geq \sum^\infty \log(1 + 1/\lambda_i) \geq \log 2 \sum_{i \geq I} 1/\lambda_i$

CTMCs: Q matrix: $0 \leq -q_{ii} < \infty$; $q_{ij} \geq 0$ $i \neq j$; $q_i := -q_{ii} = \sum_{j \neq i} q_{ij}$ | minimal: a process set to ∞ after an explosion

jump chain: $(Y_n)_{n \geq 0}$ $Y_n := X_{T_n}$ ← T_n, Z_n as above stochastic matrix Π from Q: $\pi_{ij} = \begin{cases} q_{ij}/q_i & j \neq i, q_i \neq 0 \\ 0 & j = i \\ 0 & j \neq i \quad q_i = 0 \\ 1 & j = i \quad i \text{ absorbing} \end{cases}$

$X \sim \mathrm{Markov}(\nu, Q) \Leftrightarrow (Y_n)$ is a discrete-time MC w/ initial dist ν, transition matrix Π | $\nu = \delta_k$

& cond on $Y_0 = i_0 \dots Y_{n-1} = i_{n-1}$, $Z_1 \dots Z_n$ are indep, $Z_k \sim \mathrm{Exp}(q_{i_k})$. (pf: PP for each (i,j) pair | = start from k.

SBP: $q_{ii} = -\lambda_i$, $q_{i,i+1} = \lambda_i$. $Y_n = k + n$ ← deterministic | Markov prop: same as above "phantom jump" →

stopping time: RV $0 \leq T \leq \infty$ st $\{T \leq t\}$ in terms of $(X_s)_{0 \leq s \leq t}$. – e.g. hitting times ← no proof $p_{ik}(t+s) = \sum_j \mathbb{P}(X_{t+s}, X_t = j \mid X_0 = i)$

Strong markov: T a stopping time, k st $\mathbb{P}(X_T = k) > 0$, given $T < \infty$, $X_T = k$, $(X_r)_{r \leq T}$, $(X_{T+s})_{s \geq 0}$ indep & $(X_{T+s}) \sim \mathrm{Markov}(\delta_k, Q)$.

$P(s) := (p_{ij}(s))_{i,j \in S}$, $p_{ij}(s) = \mathbb{P}(X_{T+s} = j \mid X_T = i)$ transition semigroup: $(P(t))_{t \geq 0}$: $P(0) = I$, $P(t+s) = P(t)P(s)$

backward equations: $P'(t) = QP(t)$, $P(0) = I$ & $\sum_j p_{ij}(t) = 1$ $\forall i$ gives uniq, forward eqs: $P'(t) = P(t)Q$ $P(0) = I$

$\Rightarrow P(t) = \exp(tQ) = \sum_0^\infty t^k/k!\, Q^k$. diagonalise $Q = U \Lambda U^{-1}$, $Q^k = U \Lambda^k U^{-1} \Rightarrow P(t) = U(\Sigma \cdot)U = U \exp(t\Lambda) U^{-1}$ and brute force constants

proofs: forward: finite only by first principles backward: cond. on 1st jump, × by $e^{q_i t}$, diff. $i \to j$: $\mathbb{P}_i(X_t = j$ for some $t) > 0$ with 2/max × brk/fwr

Class structure: same as discrete – open/absorbing/... iff in jump chain. nb ignore periodicity as holding times are random.

hitting times: $\tau_A^X := \inf\{t \geq 0 : X_t \in A\}$, ditto for τ_A^Y for jump chain. $\{\tau_A^X < \infty\} = \{\tau_A^Y < \infty\}$ so hitting probabilities $h_i^A := \mathbb{P}_i(\tau_A^X < \infty)$ same

⇒ hitting probs: min, non-neg sol to $h_i^A = 1$ if $i \in A$, and $h_i^A = \sum_j \pi_{ij} h_j^A$ $i \notin A$ / equiv: $\sum_j q_{ij} h_j^A = 0$ $i \notin A$. $|\{t : Y_t = i\}| \geq \sum^\infty Z_{n_k}$

i is recurrent if $\mathbb{P}(\{t : X_t = i\}$ unbdd$) = 1$, transient if $\mathbb{P}(\{t : X_t = i\}$ bdd$) = 1$, where $\{t : X_t = i\}$ is bdd if $\exists M$ st $\forall t > M$ $X_t \neq i$. ⇑

X can explode from i & X minimal ⇒ i transient recurrent ⇔ on jump chain: trans $Y \Rightarrow N = \sup_n \{Y_n = i\} < \infty$, $\{t : X_t = i\} \subseteq [0, T_{N+1})$ rec Y: inf seq N_i st $Y_{N_i} = k$

passage times: $H_i^X = \inf\{t \geq T_1 : X_t = i\}$ =: first, for seq: $H_i^{(0)} = 0$, $H_i^{(m+1)} = \inf\{t : \exists s \geq H_i^{(m)}, t \geq s, X_s \neq i, X_t = i\}$ len of $n+1$th visit

recurrent ⇔ $[q_i = 0$ or $\mathbb{P}_i(H_i < \infty) = 1] \Leftrightarrow \int_0^\infty p_{ii}(t)\, dt = \infty$ $q_i = 0$ ✓ JC rec ⇔ $\mathbb{P}_i(H_i^Y < \infty) = \mathbb{P}_i(H_i^X < \infty)$, ⇔ $\sum \pi_{ii}^{(n)} = \infty$, $\int_0^\infty p_{ii}(t) = \dots$ $\mathbb{E}_i[\sum Z_{n+1} 1_{Y_n = i}] = \sum \pi_{ii}^{(n)}$

X does not explode ⇐ state space finite OR $\sup_i q_i < \infty$ OR $X_0 = i$, i recurrent (for jump chain). | $N = \#$ children of 1 ind. $\sim \mathrm{Geo}(\frac{\mu}{\lambda + \mu})$

birth & death: $q_{ii} = -i(\lambda + \mu)$ $i \geq 1$, $q_{i,i+1} = i\lambda$, $q_{i,i-1} = i\mu$ $i \geq 1$. classes: $\{0\}$, $\mathbb{N}_{>0}$. JC = SRW ⇒ $\mathbb{P}(\text{absorb}) = \mu/\lambda$, 1 | Z = total born ← use pgf of N, Z $N \sim \mathrm{Po}(\lambda)$

SIR: λ = rate of contact, $\mathrm{Exp}(\gamma)$ = recovery time ⇒ $q_{(s,i,r)(s-1,i+1,r)} = \lambda s i$, $q_{(s,i,r)(s,i-1,r+1)} = \gamma i$ (+ diag to balance). init: $(N-m, m, 0)$, absorb: $(x, 0, N-x)$.

↑approx: w/ diff-eqs $s'(t) = -\beta s(t) i(t)$; $i'(t) = (\beta s(t) - \gamma) i(t)$; $r'(t) = \gamma i(t)$; $s(0) = 1 - \epsilon$, $i(0) = \epsilon$. $\rho = \beta/\gamma$ controls spread. ← $T := \inf\{t : I_t = 0\}$ ≤ same time for $I + G$.

^approx w/ CTMC: ignore R_t, + new G w/ $q_{(s,i,g)(s,i,g+1)} = \beta(1 - s/N)i + \beta_{g}$, $q_{(s,i,g)(s,i,g-1)} = \gamma g$. G "=" ghost individuals. S, I same as before, $G \geq 0$, $I + G \sim$ B&D proc

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Invariant dists: μ invariant ⇔ $\mu P(t) = \mu$ ⇔ $\mu Q = 0$ if μ a dist, Q non-exp, ⇒ $\mu Q = \mu P(t) Q = \mu P'(t) = \lim \frac{\mu - \mu}{h} = 0$, ⇐ $\mu P = \mu P + \int_\mu^t P'(s)$

positive recurrent: if $q_i = 0$ or $m_i = \mathbb{E}_i[H_i^X] < \infty$ ⟸⟹ $\xi_i = 1/(m_i q_i)$ class prop: split to $\mathbb{E}_i[H_j^X]$ + v.v, Geom(p) # of returns to i before j, $p = \mathbb{P}_i(H_i^X < H_j^X)$

X minimal, irreducible, pos rec conv to equilibrium: $\mathbb{P}(X_t = j) \to \xi_j$ as $t \to \infty$ Strong markov ↑, so split $\mathbb{E}_i[H_i] \dots < \infty$,

Ergodic theorem: reqs as above: $1/t \int_0^t \mathbb{1}_{\{X_s = i\}}\, ds \to \xi_i$ a.s. $\hat{\Pi}$: $\hat{\pi}_{ij} := \mu_j/\mu_i\, \pi_{ji}$: $\hat{Y} \sim \mathrm{Markov}(\mu, \hat{\Pi})$, μ invariant for $\hat{Y}$.

disc-rete [time reversal $Y \sim \mathrm{DiscMarkov}(\mu, \Pi)$, μ inv dist: fix n: $\hat{Y}_m := Y_{n-m}$ for $m \leq n$. eternal MC: $(Y_n)_{n \in \mathbb{Z}}$: uses Π on $n \geq 0$, $\hat{\Pi}$ on $n \leq 0$: then forward & rev. cond probs work.

reversible $\Pi = \hat{\Pi} \Leftrightarrow \forall i,j$ $\mu_i \pi_{ij} = \mu_j \pi_{ji}$ ← detailed balance eqs. μ solves det balance ⟹ μ invariant for Π.

cont [time reversal $\hat{X}_s$ [for $0 \leq s \leq t$] $:= X_{(t-s)-} = \lim_{r \uparrow (t-s)} X_r$. if X irred, pos-rec, min CTMC, started from inv dist μ: $\hat{X}_s$ a CTMC on $[0,t]$ w/

reversible if $Q = \hat{Q} \Leftrightarrow \mu_i q_{ij} = \mu_j q_{ji}$ = det bal ⇒ invariant!! + eternal version $\hat{Q}$: $\hat{q}_{ij} = \mu_j q_{ji} / \mu_i$

conv to eq pf: $X^{(2)} \sim \mathrm{Markov}(\mu, Q)$, $X_t = X_t^{(1)}$ $t < T$, $X_t^{(2)}$: $t \geq T = \inf t : X_t^{(1)} = X_t^{(2)}$ by Strong Markov, $\mathbb{P}(X_t = j) = \mathbb{P}(X_t' = j, T > t) + \mathbb{P}(X_t^{(2)} = j, T \leq t)$

prove diff eq: write as integral & diff that

$f(t) = 1 + K \int_0^t f(u)\, du \Rightarrow f(t) = e^{Kt}$ + need to change var in $\int$ st $m(u)$ not $m(t-s)$

Forward eqs: $p'_{ij}(t) = \sum_k p_{ik}(t) Q_{kj}$ brk: $p'_{ij} = \sum_k Q_{ik} p_{kj}(t)$ derivative of integral: of $\int_{a(x)}^{b(x)} f(x,t)\, dt$

$p_{ij}(s) = \mathbb{P}(X_{t+s} = j \mid X_t = i)$ ↑ use $\sum_j p_{ij}(t) = 1$!!!

MGF: $M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^\infty e^{tx} f(x) dx$ $M_{S_n}(t) = \prod_i M_{X_i}(t)$, $M_{aX+b} = e^{bt} M_X(at)$ $f(x, b(x)) \frac{db(x)}{dx} - f(x, a(x)) \frac{da}{dx} + \int_{a(x)}^{b(x)} \partial/\partial x\, f(x,t)\, dt$

Queues: **M/M/1**: $q_{i\,i+1}=\lambda$, $q_{i+1,i}=\mu$ $\rho:=\frac{\lambda}{\mu}$. $\rho<1$: pos rec [solve det bal] Geom$(1-\rho)$. $\rho=1$: null rec. [no valid inv dist]. $\rho>1$: transient. [JC]
busy period: $[r,s)$ st $\forall t\in[r,s), X_t\geq 1$; $X_{r-}=X_s=0$. **idle period**: same but flipped. long run busy $=\rho$. mean length busy $=1/(\mu-\lambda)$ by Ergodic
departure process: # of customers departed by time t: $D_t:=\#\{0\leq r\leq s: X_r-X_{r-}=-1\}$. obv $A\sim PP(\lambda)$ for M\M\1: **Burke's theorem**: $D\sim PP(\lambda)$ $\tilde{X}=X_{(t-s)-}$
Queue networks: X^1, X^2 M/M/1, $X^1\to X^2\to$ out, (X^1,X^2) pos rec $\Leftrightarrow$ both are | Burke prf: X in equib. X rev by det bal. $\tilde{X}\overset{d}{=}X$, $\tilde{A},\tilde{D}$ for $\tilde{X}$. $\tilde{A}\overset{d}{=}A$, $D\overset{d}{=}\tilde{A}$, as $\tilde{X}_r-\tilde{X}_{r-}=+1\Leftrightarrow X_r-X_{r-}=-1$
M/M/S: Markov: $q_{i\,i+1}=\lambda$ for $i\geq 0$ $q_{i\,i-1}=\mu\min\{i,s\}$ $\forall i\geq 1$. $\rho=\frac{\lambda}{s\mu}$: <1 pos rec [inv dist], $=1$ null rec [inv meas not dist] >1 transient [JC]
inv dist $\xi_i=K(\rho s)^i/i!$ $i\leq s$, $K\rho^i s^s/s!$ $i\geq s$. for const K [D-B] [$s=\infty\Rightarrow$ pos rec, inv dist $=$ Poisson(λ/μ), no waiting] split hs, ss=SRW
telephone exchange: truncated M/M/s: $q_{i\,i+1}=0$ for $i\geq s$. **Erlang's formula**: prob all busy $=\rho^s/s!\,/\sum_0^s \rho^j/j!$. [finite $\Rightarrow$ inv exists]

M/G/1: arrivals: $A\sim PP(\lambda)$, service $\sim G\in(0,\infty)$. $V_n:=X_{D_n}\sim$ DTMC, $\pi_{k\,k-1+m}=\mathbb{E}[e^{-\lambda G}(\lambda G)^m/m!]$ $k\geq 1, m\geq 0$ Po(λt) in time t, G's indep
$D_n:=\inf\{t>D_{n-1}: X_t-X_{t-}=-1\}$, $D_0=0$. | $\psi(\theta)=\mathbb{E}[\exp(\theta G)]=$ MGF, $\rho:=\lambda\mathbb{E}[G]$. $\mathbb{P}(N=m)=\mathbb{E}[\mathbb{P}(N=m|G)]=S$ nb $d_{0m}=d_{1m}$
inv dist: pgf is $\phi(s)=(1-\rho)(1-s)\psi(\lambda(s-1))/(\psi(\lambda(s-1))-s)$ check pgf of $\sum_{j=1}^{\infty}\xi_i\pi_{ij}$ **waiting time**: mgf: $\mathbb{E}[e^{\theta W}]=\phi(s)/\psi(\theta)$ when $\theta=\lambda(s-1)$ by pgf of $W+G$/pgf of G
G/M/1: arrivals: $A\in(0,\infty)$, service times Exp(μ). $U_n:=\#$ of other customers in queue at nth arrival: $\sim$ DTMC $a_{i\,i-j+1}=\mathbb{E}[e^{-\mu A}(\mu A)^j/j!]$
$\rho=1/\mu\mathbb{E}[A]$, $\rho<1\Rightarrow U_n$ unig inv dist $(1-q)q^k$: $q\geq 0=$ smallest sol of $s=\mathbb{E}[e^{\mu(s-1)A}]$. consider # who depart btwn arvls $i,i+1$: $\min\{i+1,$ Po$(\mu A)\}$ | $a_{i0}=1-\sum_{j=0}^{i} a_{i\,i-j+1}$ $0\leq j\leq i$
waiting time: W: $\mathbb{P}(W=0)=\mathbb{P}(\text{queue empty})=1-q$ $\mathbb{P}(W>w)=q\exp(-\mu(1-q)w)$ i.e. $W\sim$ Exp$((1-q)\mu)$ cond. on queue non-empty. $W=\sum^{N} E_i$, $E_i\sim$ Exp(μ), dist of W by MGFs.

Renewal theory

convolution: $(f*g)(t)=\int_{-\infty}^{\infty}f(t-s)g(s)\,ds$, derivative: $f'*g$ or $f*g'$

renewal process: $X_t=\#\{n\geq 1\ T_n:=\sum^n Z_k\leq t\}$, Z_k iid >0 RVs w/ dist func F. i.e. $X_0=0$.
renewal function: $m(t)=\mathbb{E}[X_t]=\mathbb{E}[\sum^\infty \mathbb{1}_{Z_k\leq t}]=\sum^\infty F_k(t)=\int_0^t\sum^\infty f^{*(k)}(s)\,ds\leq Ke^t$ $\forall t\geq 0$ by Markov on e^{-T_k}
renewal equation: $m=F+m*f$. $\tilde{X}_s=X_{T_1+s}-1$
$m(t)=\mathbb{E}[\cancel{X_t\mathbb{1}_{T_1>t}}]\,0+\int_0^t\mathbb{E}[X_t|T_1=s]f(s)ds=\int_0^t 1+m(t-s)f(s)ds$ by $\tilde{X}$.
$m=$ uniq sol that is bdd on finite intervals
renewal-type eqs: $r=H+r*f$ for $H:[0,\infty)\to\mathbb{R}$ bdd on fin intervals
uniq sol is $r:=H+H*m'$ sol: $*$ dist on t, $H*f+H*\sum f^{*k}*f=H*\sum f^{*k}$
strong law: $\mu:=\mathbb{E}[Z]\in(0,\infty)$ uniq: H bdd $\Rightarrow r$ bdd $\Rightarrow r-s=\alpha=\alpha*f^{*k}\Rightarrow\alpha=0$.
$\frac{X_t}{t}\to\frac{1}{\mu}$ a.s. as $t\to\infty$ prf: SLLN on T_n, $T_{X_t}\leq t<T_{X_t+1}$
$\frac{X_t+1}{T_{X_t+1}}\quad \frac{X_t}{X_{t+1}}\leq\frac{X_t}{t}\leq\frac{X_t}{T_{X_t}}$
CLT: $\frac{X_t-t/\mu}{\sqrt{t\sigma^2/\mu^3}}\xrightarrow{d}N(0,1)$ $a(t)=t/\mu+u\sqrt{t\sigma^2/\mu^3}$, $n(t)=\lfloor a(t)\rfloor$
$X_t\leq a(t)\Leftrightarrow X_t<n(t)\Leftrightarrow T_{n(t)}>t$
$(t-\mu n(t))/\sigma\sqrt{n(t)}=-u\,1/\ldots\to-u$.
elementary renewal theorem [no proof]
$\frac{m(t)}{t}\to\frac{1}{\mu}$ a.s as $t\to\infty$ lemma $\mathbb{E}[T_{X_t+1}]=\mu(m(t)+1)$
prf: $g(t)=\mathbb{E}[T_{X_t+1}]$, cond on T_1, $h=g/\mu-1$ sat renewal
renewal property $\forall k$ $(X_r)_{r\leq T_k}$, $(X_{T_k+s}-X_{T_k})_{s\geq 0}$ are indep $\sim X_s$

Setup: $X_t\geq k\Leftrightarrow T_k\leq t$ $\{X_t=k\}=\{T_k\leq t\}\cap\{T_{k+1}>t\}$
$F_k(t):=\mathbb{P}(T_k\leq t)$. $F_{k+1}=F_k*f=F*f^{*(k)}=f^{*k+1}$
density of $T_k=f^{*k}$ $\mathbb{P}(S+T\leq u)=(F_S*F_T)(u)$
size-biased dists: $\hat{p}_n=\frac{np_n}{\mu}$, $\hat{f}(t)=\frac{tf(t)}{\mu}$
age process $A_t=t-X_t$ **stationarity of increments**:
excess lifetime $E_t=T_{X_t+1}$ E_t doesn't dep on $t\Leftrightarrow$ $X_{t+s}-X_t\overset{d}{=}X_s$ $\forall s,t$
cont. val MC, except if Z_i's discrete

PS1: $\frac{\lfloor t\rfloor}{t}\frac{X_{\lfloor t\rfloor}}{\lfloor t\rfloor}\leq\frac{X_t}{t}\leq\frac{X_{\lceil t\rceil}}{\lceil t\rceil}\frac{\lceil t\rceil}{t}$ $\to 1$
can prove rate by differentiating.
taylor expansions: $f(x)\simeq f(x_0)+f'(x_0)(x-x_0)+\frac{f''}{2}\cdots+\frac{f^{(N)}}{N!}$
$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ $e^x=\sum_{n=0}^{\infty}x^n/n!$ $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$

delayed renewal process same except Z_1 diff dist, still indep $\leftarrow$ renewal prop still holds + for stopping times τ st $\operatorname{Im}(t)\subseteq$ jump times
$\forall t$, $\tilde{X}:=(X_{s+t}-X_t)_{s\geq 0}$ is a delayed renewal proc
but not indep of $(X_r)_{r\leq t}$
$\mathbb{P}(E_t>x)\to\frac{1}{\mu}\int_x^\infty(y-x)f(y)\,dy$ $f=$ density of Z_2, $\mu=\mathbb{E}[Z_2]$ by key-renewal-theor w/ $g(x)=1-F(x+y)$
LU: F a dist func, density $=f$, mean μ, $L=$ size-biased F
$U\sim U[0,1]$ indep of L try & int wrt U first
$\mathbb{P}(LU>x)=\frac{1}{\mu}\int_x^\infty(y-x)f(y)\,dx$ density of LU: $\frac{1}{\mu}(1-F(x))$ cond on val of L.
delayed renewal w/ $Z_1=LU$, $L=$ size-biased of Z_2 has stationary increments
$X_{t+s}-X_t\xrightarrow{d}\tilde{X}_s$ as $t\to\infty$ [$\mu<\infty$]
$(A_t,E_t)\to_d(L(1-U),LU)$
renewal theorem: $\mu<\infty$, $m(t+s)-m(t)\to s/\mu$ $\swarrow$ requires X renewal, interarrival dist is cont
key renewal theorem: $\mu<\infty$, $h:[0,\infty)\to[0,\infty)$ integ + non-incr cond of renewal theor
$(h*m')(t)=\int_0^t h(t-x)m'(x)dx\to\frac{1}{\mu}\int_0^\infty h(x)$ as $t\to\infty$.

PS2 prove stopping time: $\{S\leq t\}=\{\exists s_i\leq t\ (X_s)=m\}$
forward eqs easier: use $\sum_j p_{ij}(t)=1$
compare B&D to SBP: ghost individuals instead of dying
PS3: want $\lim_t\mathbb{P}_1(X_t=j)$ condition on first jump, assume $t>T_1$
then $\mathbb{P}_1(X_t=j, t>1, |X_1=2)=\mathbb{P}_2(X_t=j)$
(3) pos rec $\not\Leftrightarrow$ pos rec JC: e.g. $q_{nn-1}=q_{nn+1}=2^n$, π n-r
JC p-r, no inv dist for X: $q_{01}=P_{10}$ $q_{nn+1}=P_{ln}$ $q_{nn}=(1-\rho)q_n$
(5) inv dist on graph $=$ Unif dist $\Leftarrow$ by det bal
branching process: $P_n=$ pop size, $N=$ offspring dist RV, $G=\mathbb{E}[s^N]$
$\mathbb{E}[s^{P_n}]=\underbrace{G(G(\cdots G(s)))}_{n\text{ times}}$ $\mathbb{P}$(extinct) $=$ min, non-neg sol of $s=G(s)$
cont branching: $T_1, N=$ offspring
$m(t)=\mathbb{E}[X_t\mathbb{1}_{T\leq t}]+\mathbb{E}[X_t\mathbb{1}_{\{T>t\}}]$
$=\int_0^t\mathbb{E}[n\text{ copies of }X_{t-u}\mathbb{1}_{N\geq 1}]e^{-u}du+e^{-t}$ $\Rightarrow m(t)=e^{(N-1)t}$
$\mathbb{E}[N]\mathbb{E}[X_{t-u}]$ by random-sum

Geom: $(1-\rho)\rho^i\leftarrow$ not $i-1$, here.
Random sum: X_i iid, indep of $N\geq 0$ $\mathbb{E}[\sum_i^N X_i]=\mathbb{E}[N]\,\mathbb{E}[X_i]$ general: just write F_1*F_2
Poisson dist: $\mathbb{P}(X=n)=e^{-\lambda}\lambda^n/n!$ mean $=\lambda$ Exp: $f(t)=\lambda e^{-\lambda t}$, $\mathbb{P}(X>t)=e^{-\lambda t}$, $\mathbb{E}[X]=1/\lambda$ Exp$(\lambda)+$Exp(λ) ind $=$ Gamma$(2,\lambda)$